Local rigid cohomology of weighted homogeneous hypersurface singularities

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Summary

The goal of this thesis is to study a certain invariant of isolated singularities over a base field k of positive characteristic. This invariant is called the *local rigid cohomology*. For a singular point $x \in X$ on a k-scheme, the i-th local rigid cohomology is defined as $H^i_{rig,\{x\}}(X)$, the i-th rigid cohomology of X with supports in the subset $\{x\}$.

In chapter 2 we show that the local rigid cohomology is indeed an invariant. That is: if $x' \in X'$ and $x \in X$ are contact-equivalent singularities on k-schemes, then the local rigid cohomology spaces $H^\bullet_{rig,\{x\}}(X)$ and $H^\bullet_{rig,\{x'\}}(X')$ are isomorphic. The isomorphism that we construct is moreover compatible with the Frobenius action on rigid cohomology.

In chapters 3 and 4 we focus our attention on weighted homogeneous hypersurface singularities. Our goal in chapter 3 is to show that for such a singularity, the local rigid cohomology may be identified with the $G(\underline{w})$-invariants of a certain rigid cohomology space $H^\bullet_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$. Here $\widetilde{S}_\infty \subset \mathbb{P}^{n-1}_k$ is a smooth projective hypersurface, and $G(\underline{w})$ is a certain finite group acting on the rigid cohomology of its complement.

It is known that the rigid cohomology of a smooth projective hypersurface is amenable to direct computation. Indeed, an algorithm by Abbott, Kedlaya and Roe allows one to approximate the Frobenius on such a rigid cohomology space. In chapter 4 we will modify this algorithm to deal with the $G(\underline{w})$-invariant part of cohomology. The modified algorithm can be formulated entirely in terms of weighted homogeneous polynomials, which seems natural for our applications.

Chapter 5 is a collection of conjectures and open problems that are related to the earlier chapters.

Zusammenfassung

Das Ziel dieser Dissertation ist die Erforschung einer gewissen Invariante von Singularitäten über einem Grundkörper k von positiver Charakteristik. Sei $x \in X$ ein singulärer Punkt auf einem k-Schema. Dann ist die lokale rigide Kohomologie im Grad i definiert als $H^i_{rig,\{x\}}(X)$, also als die rigide Kohomologie von X mit Träger in der Teilmenge $\{x\}$.

In Kapitel 2 zeigen wir, dass die lokale rigide Kohomologie tatsächlich eine Invariante ist. Das heißt: Sind $x' \in X'$ und $x \in X$ kontaktäquivalente singuläre Punkte auf k-Schemata, dann sind die Vektorräume $H^\bullet_{rig,\{x\}}(X)$ und $H^\bullet_{rig,\{x'\}}(X')$ zueinander isomorph. Dieser Isomorphismus ist kompatibel mit der Wirkung des Frobenius auf der rigiden Kohomologie.

In den Kapiteln 3 und 4 beschäftigen wir uns mit gewichtet homogenen Singularitäten von Hyperflächen. Der Hauptsatz des dritten Kapitels besagt, dass die lokale rigide Kohomologie einer solchen Singularität isomorph ist zu dem $G(\underline{w})$-invarianten Teil von $H^\bullet_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$. Hier bezeichnet $\widetilde{S}_\infty \subset \mathbb{P}^{n-1}_k$ eine gewisse glatte projektive Hyperfläche und $G(\underline{w})$ ist eine endliche Gruppe, die auf der rigiden Kohomologie des Komplements wirkt.

Dank einem Algorithmus von Abbott, Kedlaya und Roe ist es möglich, den Frobenius-Automorphismus auf $H^\bullet_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$ annähernd zu berechnen. In Kapitel 4 formulieren wir eine Anpassung dieses Algorithmus, mithilfe derer Berechnungen auf dem $G(\underline{w})$-invarianten Teil gemacht werden können. Der angepasste Algorithmus kann vollständig mithilfe gewichtet homogener Polynome formuliert werden, was für unsere Anwendungen sehr natürlich scheint.

In Kapitel 5 formulieren wir einige Vermutungen und offene Probleme, die mit den Ergebnissen der früheren Kapitel zusammenhängen.

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Contents

Chapter 1

Introduction

1.1 Overview of results

The goal of this thesis is to study a certain invariant of singularities on algebraic varieties in positive characteristic, namely the *local rigid cohomology.* In this section we will give an intuitive overview of this invariant, as well as an informal description of the results that will be proved in later chapters.

1.1.1 Local cohomology of complex singularities

Let us start with a very brief overview of the classical theory of singularities over the complex numbers. For this, consider an algebraic variety X over the field of complex numbers $\mathbb{C}$. A point $x \in X$ is said to be *singular* if the local ring $\mathcal{O}_{X,x}$ is *not* regular. A singular point $x \in X$ is said to be *isolated* if $\mathcal{O}_{X,y}$ is regular for all $y \neq x$ in some Zariski-open neighbourhood of x. For such an isolated singularity it is known that its "shape" is completely determined by the ring $\widehat{\mathcal{O}}_{X,x}$, the completion of the local ring of X at x. Therefore two singularities $x \in X$ and $x' \in X'$ should be treated as equivalent if their completed local rings $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{X',x'}$ are isomorphic. This relation is also called *contact equivalence.*

So far all the definitions are purely algebraic. But in order to define the *local cohomology* of a complex singularity, one really needs the complex structure. Indeed, the set of $\mathbb{C}$-points $X(\mathbb{C})$ can be given the structure of a complex analytic space [Ser56]. In particular, $X(\mathbb{C})$ is a Hausdorff space. This topological space may also be interpreted as a refinement of the Zariski topology on X, in the sense that there is a canonical continuous map $X(\mathbb{C}) \to X$. The i-th local cohomology at x is then defined as $H^i_{\{x\}}(X, \mathbb{C})$. This is the i-th Betti cohomology space of $X(\mathbb{C})$, with complex coefficients and with supports in the set $\{x\}$[1]. By definition, this object only depends on a small neighbourhood of x in $X(\mathbb{C})$.

[1]The notation may be a bit confusing, but it is standard.

Now assume that $x' \in X'$ is another singularity that is contact equivalent to $x \in X$. Then it can be shown that there exists a diffeomorphism $\varphi\colon U \xrightarrow{\sim} V$, where U (resp. V) is a small neighbourhood of $x \in X(\mathbb{C})$ (resp. of $x' \in X'(\mathbb{C})$), and such that $\varphi(x) = x'$. This is a consequence of Artin approximation, see [Art68, Corollary 1.6] for details. The diffeomorphism φ then induces an isomorphism

$$H^i_{\{x'\}}(X', \mathbb{C}) \xrightarrow{\sim} H^i_{\{x\}}(X, \mathbb{C}). \tag{1.1.1}$$

In other words: the local cohomology of a singularity is preserved under contact equivalence. Therefore it is an *invariant* of the singularity.

It is also known that the local cohomology of a complex singularity carries a mixed Hodge structure [Ste83]. This additional structure is also preserved by the isomorphism (1.1.1). The mixed Hodge structure is really what makes the local cohomology into an interesting invariant. After all, a finite-dimensional vector space is completely determined by its dimension. Without the mixed Hodge structure, the local cohomology would be nothing but a sequence of Betti numbers.

1.1.2 Generalizing to positive characteristic

The notions of a singularity and of contact equivalence can readily be extended to algebraic varieties over any base field. The notion of local cohomology does not extend so easily, because the complex structure is no longer there. Indeed, it is known that taking the classical cohomology with respect to the Zariski topology does *not* yield a satisfactory theory. Instead one should consider other, more abstract, cohomology theories.

Rigid cohomology is a cohomology theory that was specifically designed for algebraic varieties over a base field of positive characteristic. Indeed, consider a variety X over (say) a prime field $\mathbb{F}_p$. Then for each $i \geq 0$ the rigid cohomology space $H^i_{rig}(X)$ is a finite-dimensional vector space over the p-adic field $\mathbb{Q}_p$. For this reason, rigid cohomology is also known as *p-adic cohomology*. It can be seen as an alternative to ℓ-adic (étale) cohomology, which takes values in a field $\mathbb{Q}_\ell$ for a prime $\ell \neq p$.

For a closed subset $Z \subset X$ there is also the notion of *rigid cohomology with supports*. The i-th rigid cohomology of X with supports in Z is denoted by $H^i_{rig,Z}(X)$.

Now consider again an isolated singular point x on the $\mathbb{F}_p$-variety X. Then the i-th *local rigid cohomology* of X at x is defined as $H^i_{rig,\{x\}}(X)$, the i-th rigid cohomology of X with supports in $\{x\}$. This is the object that will be studied in chapters 2, 3 and 4.

The local rigid cohomology comes with an additional structure called the *action of Frobenius* (often shortened to just *Frobenius*). In fact, every rigid cohomology space $H^i_{rig,Z}(X)$ for $i \geq 0$ and $Z \subset X$ closed is equipped with such a structure. Concretely, the Frobenius action is just an invertible linear map on $H^i_{rig,Z}(X)$. We will elaborate on its definition in paragraph 2.1.3. The local rigid cohomology $H^i_{rig,\{x\}}(X)$ of a singular point $x \in X$ should always be considered together with its Frobenius action. In this sense, the local rigid cohomology is more than just a sequence of Betti numbers. In a way, the Frobenius action is comparable to the mixed Hodge structure that exists on the Betti cohomology over $\mathbb{C}$.

1.1.3 The results of this thesis

One natural question that now arises is: Is the local rigid cohomology also an invariant for the singularity $x \in X$, as was the case for the local cohomology over $\mathbb{C}$? The answer is *yes*, and this is the main result of chapter 2.

The fact that the local cohomology over $\mathbb{C}$ is an invariant is a direct corollary of Artin approximation. Over a base field of positive characteristic one can use another form of Artin approximation to conclude that two contact equivalent points have isomorphic étale neighbourhoods. See paragraph 2.1.1 for details. The main difficulty in chapter 2 is then to prove that an isomorphism of étale neighbourhoods induces an isomorphism on the local rigid cohomology. This is precisely the content of theorem 2.1.8.

In chapters 3 and 4 we will focus on the special class of *weighted homogeneous hypersurface singularities*. These are the singularities that are contact equivalent to the origin of an affine hypersurface $Y = Z(g) \subset \mathbb{A}^n_k$, where $g \in k[x_1, \ldots, x_n]$ is a weighted homogeneous polynomial. See paragraph 3.1.1 for more detailed definitions.

The polynomial g is also called the *normal form* of the weighted homogeneous singularity, and in general it only appears after a suitable change of local coordinates. But by the theorem of chapter 2 we know that the local rigid cohomology of a weighted homogeneous singularity is isomorphic to $H^\bullet_{rig,\{0\}}(Y)$, with Y as above.

The general definition of rigid cohomology is not very practical to work with. However, in several situations it is possible to give a simpler characterization of p-adic cohomology. In particular this is the case for the rigid cohomology $H^\bullet_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$ of the complement of a smooth projective hypersurface $\widetilde{S}_\infty \subset \mathbb{P}^{n-1}_k$. We will come back to this in section 1.2 below.

The goal of chapter 3 is to show that for a weighted homogeneous hypersurface singularity $Y = Z(g) \subset \mathbb{A}^n_k$, there is an isomorphism

$$H^n_{rig,\{0\}}(Y) \xrightarrow{\sim} H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}. \tag{1.1.2}$$

As the notation suggests, $\widetilde{S}_\infty \subset \mathbb{P}^{n-1}_k$ is a smooth projective hypersurface. It can be defined from the local equation g, see definition 3.1.5. The rigid cohomology $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$ has an action from a certain finite group $G(\underline{w})$, and the local rigid cohomology may be identified with its invariants. So the isomorphism (1.1.2) says that for the local rigid cohomology of a weighted homogeneous singularity, we are "almost" in a situation where rigid cohomology reduces to a simpler theory. The complete formal statement is given in theorem 3.1.11.

This result should also be seen as parallel to certain classical complex-analytic results. There are however some differences with the analytic setting. Over $\mathbb{C}$ one would typically consider the *weighted projective* hypersurface that is defined by the equation g. One major technical problem of weighted projective spaces is that they have singularities. Although the Betti cohomology of weighted projective hypersurfaces is well understood, this does not seem to be the case for rigid cohomology. For this reason we only consider (non-weighted) projective hypersurfaces, at the cost of imposing a slightly restrictive assumption on g (see definition 3.1.7).

We will recall the classical complex results in paragraph 3.4.1. The reason is that these results can (somewhat surprisingly) still be used in our proofs for rigid cohomology. In the same paragraph we also give more detailed explanations about the technical difficulties that are encountered in rigid cohomology. See in particular remark 3.4.2.

Chapter 4 deals with explicit computations on the $G(\underline{w})$-invariant cohomology space that is on the right-hand side of the isomorphism (1.1.2). Here we draw inspiration from the algorithm of Abbott, Kedlaya and Roe, which is found in [AKR11]. In this paper, the authors give an algorithm to approximate the Frobenius action on $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$, where $\widetilde{S}_\infty \subset \mathbb{P}^{n-1}_k$ is a smooth projective hypersurface. We will give a quick overview of this algorithm in paragraph 1.2.3 below.

In chapter 4 we will modify the AKR algorithm to deal with the $G(\underline{w})$-invariants that appear on the right-hand side of equation (1.1.2).

This in itself is not so difficult. In paragraph 4.2.1 we show that the AKR algorithm also works for the $G(\underline{w})$-invariant subspace, provided that the basis is of a particular form. However, this solution is not entirely satisfactory, since the algorithm to approximate the Frobenius on $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}$ is still "embedded" into the classical AKR algorithm. In other words: we are carrying some additional information that does not relate to the $G(\underline{w})$-invariants.

Paragraphs 4.2.3 and 4.2.4 contain the most important results of chapter 4. Here we reformulate the AKR algorithm in such a way that it *only* considers the $G(\underline{w})$-invariant part of the cohomology. This algorithm can be formulated entirely in terms of weighted homogeneous polynomials, which seems more natural as well.

In the end it turns out that this modified algorithm was already known, but with a different interpretation. See paragraph 4.2.6 for details. So our results from chapter 4 can be reformulated by saying that the previously known algorithm is related to rigid cohomology.

Chapter 5 is essentially a collection of conjectures and open problems that are related to the earlier chapters.

We start this chapter by giving an overview of some of the technical difficulties that we encountered in chapter 3. The goal is to give a more complete side-by-side comparison of the theory in characteristic zero and the theory over a base field of positive characteristic.

In sections 5.2 and 5.3 we formulate some conjectures about the rigid cohomology $H^\bullet_{rig}(\mathbb{P}^n_k \setminus X)$, where $X \subset \mathbb{P}^n_k$ is a *singular* hypersurface. We also explore the idea that these conjectures should come from certain relations with the local cohomology spaces $H^\bullet_{rig,\{x\}}(X)$, where $x \in X$ is a singular point. Such relations sometimes yield a proof in the case where all the singularities of X are weighted homogeneous, see for example corollary 5.3.6.

1.2 Review of p-adic cohomology

In the remainder of this introductory chapter we give a review of a few important definitions and results from p-adic cohomology. As mentioned in paragraph 1.1.3, there are situations in which rigid cohomology reduces to a somewhat simpler theory. These special cases will be covered in paragraphs 1.2.2 and 1.2.3. The last two paragraphs deal with the general definition of rigid cohomology. But first we fix a few notations and technical assumptions.

1.2.1 Conventions and notations

Throughout this thesis k denotes a perfect field of characteristic $p > 0$. We also fix a Frobenius map $x \mapsto x^q$ on k, where $q = p^r$ for some $r \geq 1$.

In addition we let K denote a field of characteristic zero, complete w.r.t. a discrete valuation, with valuation ring $(\mathcal{V}, \pi)$ whose residue field $\mathcal{V}/(\pi)$ is isomorphic to the field k above. Moreover we assume that the Frobenius map on k admits an isometric lift $\sigma\colon K \to K$. We fix such a lift for the remainder of this thesis.

Every scheme X will be assumed to be defined over k, unless we specify a different base. We will tacitly assume all k-schemes to be reduced, of finite type and separated. Morphisms between k-schemes are assumed to be defined over k. Every closed subset $Z \subset X$ will be equipped with the reduced subscheme structure.

We will only consider formal schemes defined over $\mathrm{Spf}(\mathcal{V})$. Such a formal scheme will be assumed to be separated and topologically of finite type over $\mathrm{Spf}(\mathcal{V})$. See [Nic08] for the relevant definitions.

Note that in general we don't assume k to be finite. Indeed, the results in chapter 2 hold for any perfect base field k of positive characteristic. In chapter 3 we will start to work over a finite base field, although this is not strictly necessary until chapter 4.

We quickly recall the typical choices for K and σ when k is algebraic over its prime field.

- If $k = \mathbb{F}_p$ for some prime p then we may take $K = \mathbb{Q}_p$. Then $\mathcal{V} = \mathbb{Z}_p$ and $p \in \mathbb{Z}_p$ is a local parameter. Every Frobenius map $x \mapsto x^{p^r}$ is equal to the identity on k, therefore $\sigma = \mathrm{Id}_K$.

- Next consider the case where $k = \mathbb{F}_{p^s}$. In this situation one usually takes $\mathcal{V} = W(k)$, the ring of Witt vectors over k. We take $K = \mathrm{Frac}\,\mathcal{V}$. It is a well-known fact that K is a finite unramified extension of $\mathbb{Q}_p$. This implies that $p \in \mathbb{Z}_p \subset \mathcal{V}$ is again a local parameter. It is also known that the map

$$\begin{array}{rcl} \mathrm{Gal}(K/\mathbb{Q}_p) & \to & \mathrm{Gal}(k/\mathbb{F}_p) \\ \sigma & \mapsto & \sigma_{|\mathcal{V}} \bmod p \end{array}$$

is an isomorphism of groups. Therefore we have a unique lift σ of the Frobenius map $x \mapsto x^{p^r}$ on k. If $s \mid r$ then of course $\sigma = \mathrm{Id}_K$.

- Finally we consider the algebraic closure of a finite field. If $k = \overline{\mathbb{F}_p}$ then we may take K to be the completion of $\mathbb{Q}_p^{unr}$, the maximal unramified extension of $\mathbb{Q}_p$. Again there is an isomorphism

$$\mathrm{Gal}(\mathbb{Q}_p^{unr}/\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Gal}(k/\mathbb{F}_p),$$

from which we find a unique Frobenius σ on K.

The proofs for the statements above can be found in [Ser79]. It is worth noting that a suitable field K exists for a general base field k, thanks to [Ser79, Theorem II.3].

1.2.2 Monsky-Washnitzer cohomology

Monsky and Washnitzer were the first to define a p-adic arithmetic cohomology theory [MW68]. This theory is limited to smooth affine schemes. It is however easy to understand because it is quite close to traditional de Rham cohomology (closed forms modulo exact forms). Monsky-Washnitzer cohomology is also important because it can be algorithmically computed for certain smooth affine schemes. We will explain this further in the next paragraph.

We will see in paragraph 1.2.4 that Monsky-Washnitzer cohomology is a special case of rigid cohomology. This is a more general p-adic cohomology theory that also works for singular schemes. To *define* the invariants of a singularity that we mentioned in paragraph 1.1.2 we will need to use rigid cohomology. However, to *compute* these invariants we will always reduce to a situation where we can use Monsky-Washnitzer cohomology.

Consider a smooth affine scheme $X = \operatorname{Spec} \overline{A}$. It turns out that working over a field of characteristic $p > 0$ has the disadvantage that many closed forms that "should be" integrable are not. For example, if we take $\overline{A} = k[x]$ then the forms $x^{p^n-1}\,dx$ for $n \geq 1$ are all closed but not exact. Indeed, if we try to integrate these forms we obtain

$$\int x^{p^n-1}\,dx \text{ “ = ” } \frac{1}{p^n}\, x^{p^n},$$

which is not meaningful in characteristic p. So the classical definition of de Rham cohomology does not yield a good cohomology theory in characteristic p. The idea of Monsky and Washnitzer is to find a $\mathcal{V}$-algebra B such that $B \otimes_{\mathcal{V}} k \cong \overline{A}$ and then to consider differential forms on B. Since $\mathcal{V}$ is a domain of characteristic zero, this should resolve the difficulties that arise in characteristic p. The problem is that there may be many choices for the lifted algebra B. The right choice for such a lift is called the *weak completion* of $\overline{A}$. In [MW68] this lift is denoted by A.

The weak completion can be easily understood in terms of the *dagger operator*, as explained in [vdP86]. This operator maps a finitely generated $\mathcal{V}$-algebra B to a weakly complete $\mathcal{V}$-algebra $B^\dagger$. Concretely, $\mathcal{V}[x_1, \ldots, x_n]^\dagger$ is given by the algebra of power series in the variables $x_1, \ldots, x_n$ with coefficients in $\mathcal{V}$ whose radius of convergence is strictly greater than 1. The weak completion A of $\overline{A}$ is isomorphic to the algebra $B^\dagger$ where B is any *smooth* lift of $\overline{A}$ to $\mathcal{V}$ [vdP86, Theorem 2.4.4]. The existence of such a smooth lift is guaranteed by [Elk73, Théorème 6]. Note that while the weak completion A is uniquely defined, a smooth lift B need *not* be unique.

Monsky-Washnitzer cohomology is then defined in terms of the modules of *continuous* differential forms on A. These modules form a complex $D^\bullet(A)$. The precise definition is given in section 4 of [MW68]. Note however that $D^i(A) \cong \Omega^i_B \otimes_B B^\dagger$ with B as above and where Ω^i_B is the canonical module of i-forms on B. This is an easy consequence of the alternative definition [vdP86, Definition 2.3]. The Monsky-Washnitzer cohomology is essentially defined as the cohomology of this complex:

$$H^i_{MW}(X) \coloneqq H^i\left(D^\bullet(A) \otimes_{\mathcal{V}} K\right)$$

with K the fraction field of $\mathcal{V}$.

Suppose that $X' = \operatorname{Spec} \overline{A}'$ is another smooth affine k-scheme and that we have a morphism $f \colon X' \to X$ that is given by the Spec of $\varphi \colon \overline{A} \to \overline{A}'$. Then it can be shown that there exists a lift $A \to A'$ on the weak completions. Moreover, the induced map $D^\bullet(A) \otimes K \to D^\bullet(A') \otimes K$ is unique modulo a chain homotopy [vdP86, Theorem 2.4.4], which means that it induces a well-defined map $H^\bullet_{MW}(X) \to H^\bullet_{MW}(X')$. In this way the Monsky-Washnitzer cohomology $H^\bullet_{MW}$ becomes a functor.

If we take $k = \mathbb{F}_q$ and $\varphi^{(q)} \colon \overline{A} \to \overline{A}$ the q-power Frobenius on $\overline{A}$ then we obtain the *Frobenius map* $\mathrm{Fr}^{(q)} \colon H^\bullet_{MW}(X) \to H^\bullet_{MW}(X)$. The *trace formula* gives a relation between the Frobenius maps on $H^\bullet_{MW}(X)$ and the number of rational points on X. For a smooth affine scheme X of dimension n we have:

$$|X(k)| = \sum_{i=0}^{n} (-1)^i \operatorname{trace}\left(q^n \operatorname{Fr}^{-1} \mid H^i_{MW}(X)\right). \tag{1.2.1}$$

There is also a formula that relates the zeta function of X to the cohomology:

$$Z(X,T) = \prod_{i=0}^{n} \det\left(1 - q^n \operatorname{Fr}^{-1} T \mid H^i_{MW}(X)\right)^{(-1)^{i+1}}. \tag{1.2.2}$$

See [vdP86] for the proofs of these results. In formulas (1.2.1) and (1.2.2) the symbols $\left(\mathrm{Fr} \mid H^i_{MW}(X)\right)$ are used as abbreviations for the q-power Frobenius $\mathrm{Fr}^{(q)}$ on the i-th cohomology space $H^i_{MW}(X)$.

Remark 1.2.1. Assume again that $k = \mathbb{F}_q$ with $q = p^s$ for some prime p and $s \geq 1$. We could also have considered the p^r-power Frobenius $\varphi^{(p^r)}$ on $\overline{A}$ for $r \neq s$. The main difference is that if $s \nmid r$ then the p^r-power Frobenius on k is not the identity. Therefore $\varphi^{(p^r)}$ is *not* a morphism of k-algebras and the methods above cannot be used to define a corresponding Frobenius map on cohomology. There is however a more general construction that still allows one to define a p^r-power Frobenius map $\mathrm{Fr}^{(p^r)}$ on $H^\bullet_{MW}(X)$. See paragraph 8.3 in [LS07] for details. Note that this Frobenius map is σ-linear. One should be careful that the formulas (1.2.1) and (1.2.2) *only* hold for the q-power Frobenius over $\mathbb{F}_q$, for which $\sigma = \mathrm{Id}$.

1.2.3 The AKR algorithm

Although the definition of Monsky-Washnitzer cohomology is rather abstract, there are some situations in which it becomes amenable to computation. Specifically, this means that one can determine a basis for a given Monsky-Washnitzer cohomology space and use this basis to compute an *approximation* of the matrix of the Frobenius action with respect to this basis. For the complement of smooth projective hypersurface this has been worked out in [AKR11].

It should be clear that one cannot expect to do better than to find an approximation of the Frobenius matrix, since it has entries in K. The field K is an extension of a p-adic field and its elements can in general not be written down exactly. However, it turns out that such an approximation can already be enough to *exactly* compute certain objects related to p-adic cohomology.

The (partial) computability of Monsky-Washnitzer cohomology relies on the following theorem due to Baldassarri and Chiarellotto.

Theorem 1.2.2. *Let $\mathcal{Y}$ be a smooth proper $\mathcal{V}$-scheme and $\mathcal{X} \subset \mathcal{Y}$ a strict normal crossings divisor. Also assume that the complement $\mathcal{Y} \setminus \mathcal{X}$ is affine. Denote by $X \subset Y$ and $\mathcal{X}_K \subset \mathcal{Y}_K$ the fibers over k resp. over K. Then for all $i \geq 0$ there is a canonical isomorphism of K-vector spaces*

$$H^i_{dR}(\mathcal{Y}_K \setminus \mathcal{X}_K) \xrightarrow{\sim} H^i_{MW}(Y \setminus X) \tag{1.2.3}$$

where the left-hand side is the i-th algebraic de Rham cohomology space and the right-hand side is the i-th Monsky-Washnitzer cohomology.

Proof. This is a special case of the main result of [BC94]. $\square$

The result of theorem 1.2.2 becomes even more concrete at the level of algebras. Write $Y \setminus X = \operatorname{Spec} \overline{A}$ and $\mathcal{Y} \setminus \mathcal{X} = \operatorname{Spec} B$, with $\overline{A}$ a smooth k-algebra and B a smooth $\mathcal{V}$-algebra. Then the weak completion A of $\overline{A}$ is isomorphic to $B^\dagger$ and we have a *specialization map*

$$H^i(\Omega^\bullet_B \otimes K) \longrightarrow H^i(\Omega^\bullet_B \otimes B^\dagger \otimes K) \tag{1.2.4}$$

that is induced by the arrow $\Omega^\bullet_B \otimes K \hookrightarrow \Omega^\bullet_B \otimes B^\dagger \otimes K$. Under the conditions of theorem 1.2.2, this specialization map is precisely the isomorphism (1.2.3).

This isomorphism can be used to transfer the Frobenius action from the Monsky-Washnitzer cohomology to $H^i(\Omega^\bullet_B \otimes K)$. To be more precise, assume that the algebra B has a presentation

$$B = \frac{\mathcal{V}[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)},$$

which gives us a Frobenius lift

$$F \colon B^\dagger \to B^\dagger \colon x_i \mapsto \sum_{j=1}^{\infty} \alpha_{ij}(x_1, \ldots, x_n).$$

Then F induces a map F^* on $D^\bullet(B^\dagger) \otimes K \cong \Omega^\bullet_B \otimes B^\dagger \otimes K$. If $\omega \in \Omega^i_B \otimes K$ is a closed differential form that represents an element of $H^i(\Omega^\bullet_B \otimes K)$, the image $F^*\omega$ can be written as a $B^\dagger$-linear combination

$$\sum_{m_1 < \ldots < m_i} b_{m_1,\ldots,m_i}\, dx_{m_1} \wedge \ldots \wedge dx_{m_i}. \tag{1.2.5}$$

By the isomorphism (1.2.3) the resulting differential form can be thought of as an element of $H^i(\Omega^\bullet_B \otimes K)$. If we choose a basis $\omega_1, \ldots, \omega_\delta$ for $H^i(\Omega^\bullet_B \otimes K)$

then the element (1.2.5) corresponds to a certain linear combination

$$\lambda_1\omega_1 + \ldots + \lambda_\delta\omega_\delta. \tag{1.2.6}$$

These observations are not yet sufficient to give an algorithm that approximates the Frobenius action on $H^i_{MW}(Y \setminus X)$. All the steps need to be made *effective*. This means that one needs:

- A concrete basis $\omega_1, \ldots, \omega_\delta$ for $H^i_{dR}(\mathcal{Y}_K \setminus \mathcal{X}_K)$.
- A concrete representation of the elements $b_{m_1,\ldots,m_i} \in B^\dagger$ in equation (1.2.5), together with a way to *truncate* these elements in order to obtain *polynomials* as coefficients.
- A procedure to transform the *truncated* sum (1.2.5) into an *approximate* representation (1.2.6).
- Precision estimates on the resulting approximate coefficients $\lambda_1, \ldots, \lambda_\delta$.

It should be clear that such a level of detail can only be achieved for certain concrete families of varieties $\mathcal{X} \subset \mathcal{Y}$.

The first algorithm of this type was developed by Kedlaya in [Ked01]. This paper deals with the case where Y is a hyperelliptic curve over a finite field and $X \subset Y$ is the set of Weierstraß points (plus the point at infinity).

The algorithm contained in [Ked01] can be used to *exactly* calculate the characteristic polynomial of the Frobenius on $H^1_{MW}(Y \setminus X)$. Indeed, this characteristic polynomial has integer coefficients. In paragraph 3 of [Ked01] it is shown that the complex norms of these coefficients are bounded as a result of the Riemann hypothesis for smooth curves. This results in an explicit formula for the p-adic precision on the elements $\lambda_1, \ldots, \lambda_\delta$ that is needed to exactly recover the characteristic polynomial. The most subtle result in [Ked01] is that the precision can be controlled by the level of truncation that is chosen in the sum (1.2.5). From this one obtains an algorithm that (exactly) computes the zeta function of a hyperelliptic curve.

After Kedlaya's paper many variations and improvements have appeared, for various classes of curves.

In [AKR11] a similar method has been developed for the case where $Y = \mathbb{P}^{n-1}_k$ with k a finite field and $X = Z(\widetilde{g}) \subset \mathbb{P}^{n-1}_k$ is any smooth projective hypersurface. The method of Abbott, Kedlaya and Roe relies on the description of the de Rham cohomology $H^\bullet_{dR}(\mathbb{P}^{n-1}_K \setminus \mathcal{X}_K)$ that is due to Griffiths [Gri69]. This approach is a bit different than in [Ked01] because the polynomial $\widetilde{g}$ defining X doesn't belong to a fixed family of equations.

The theory of Griffiths

We briefly recall the content of Griffiths' classical paper [Gri69]. Let $\widetilde{\mathcal{G}} \in \mathcal{V}[x_1, \ldots, x_n]$ with $n \geq 2$ be a homogeneous polynomial of degree d. Assume moreover that the projective hypersurface

$$\mathcal{X}_K = Z(\widetilde{\mathcal{G}}) \subset \mathbb{P}^{n-1}_K$$

is smooth.

In this situation it is well-known that $H^0_{dR}(\mathbb{P}^{n-1}_K \setminus \mathcal{X}_K)$ is one-dimensional while $H^i_{dR}(\mathbb{P}^{n-1}_K \setminus \mathcal{X}_K) = 0$ for $0 < i < n-1$. See paragraph 3.5.1 for more details.

The nonzero differential $(n-1)$-forms on the affine complement $\mathbb{P}^{n-1}_K \setminus \mathcal{X}_K$ can be written as

$$\frac{A\,\Omega}{\widetilde{\mathcal{G}}^t}, \tag{1.2.7}$$

where

$$\Omega = \sum_{i=1}^{n} (-1)^{i-1} x_i \cdot dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$$

and $A \in K[x_1, \ldots, x_n]$ is a homogeneous polynomial of degree $td - n$. The integer $t \geq 1$ is referred to as the *pole order.* Since all $(n-1)$-forms are closed, the algebraic de Rham cohomology $H^{n-1}_{dR}(\mathbb{P}^{n-1}_K \setminus \mathcal{X}_K)$ is equal to the K-space generated by the forms (1.2.7), modulo the forms that are exact.

A differential form that is of the shape (1.2.7) is completely determined by the polynomial A. Given two homogeneous polynomials A and A' of degrees $td-n$ resp. $t'd-n$, it is natural to ask whether the associated forms determine the same cohomology class in $H^{n-1}_{dR}(\mathbb{P}^{n-1}_K \setminus \mathcal{X}_K)$. To answer this question one should understand all the relations between the differential forms (1.2.7). This amounts to the same thing as having an explicit set of generators for the space of exact forms. Such generators are provided by the following result of Griffiths.

Proposition 1.2.3. *Write the de Rham cohomology space $H^{n-1}_{dR}(\mathbb{P}^{n-1}_K \setminus \mathcal{X}_K)$ as Z^{n-1}/B^{n-1}, where Z^{n-1} is the K-vector space generated by the forms (1.2.7). The subspace B^{n-1} of exact forms is generated by the elements*

$$\frac{B \cdot (\partial_i \widetilde{\mathcal{G}})\,\Omega}{\widetilde{\mathcal{G}}^t} - (t-1)^{-1} \frac{\partial_i B\,\Omega}{\widetilde{\mathcal{G}}^{t-1}} \tag{1.2.8}$$

for any $1 \leq i \leq n$, any $t \geq 2$ and any homogeneous polynomial B of degree $(t-1)d - n + 1$.

Proof. This is a consequence of [Gri69, Theorem 2.9]. Also see paragraph 4 of the same paper. □

As a corollary one can show that a form of the shape (1.2.7) is cohomologous to a differential form of pole order $t-1$ if and only if A belongs to the Jacobian ideal

$$J = (\partial_1 \widetilde{\mathcal{G}}, \ldots, \partial_n \widetilde{\mathcal{G}}).$$

See [Gri69, Proposition 4.6]. If we can write $A = B_1 \cdot (\partial_1 \widetilde{\mathcal{G}}) + \ldots + B_n \cdot (\partial_n \widetilde{\mathcal{G}})$ then the relations (1.2.8) give an explicit formula for a form

$$\frac{A'\,\Omega}{\widetilde{\mathcal{G}}^{t-1}}$$

to which (1.2.7) is cohomologous. This procedure is called *reduction of the pole order*.

By a result of Macaulay it is known that any homogeneous polynomial A corresponding to a pole order $t > n-1$ belongs to the Jacobian ideal J. Therefore every $(n-1)$-form on $\mathbb{P}_K^{n-1} \setminus \mathcal{X}_K$ is cohomologous to a form with pole order at most $n-1$.

The facts above can be used to compute a certain decomposition, called the *reduced form* or the *Griffiths-Dwork reduction*, of a differential $(n-1)$-form. Indeed, by repeatedly computing the remainder $r = A$ rem J and applying pole order reduction to $A - r$, one obtains the representation

$$\frac{A\,\Omega}{\widetilde{\mathcal{G}}^t} = \sum_{\alpha=1}^{n-1} \frac{r_\alpha\,\Omega}{\widetilde{\mathcal{G}}^\alpha}. \tag{1.2.9}$$

This reduced form enjoys the following property: two homogeneous polynomials A and A' represent the same cohomology class if and only if their reduced forms are the same. See [BLS13, Algorithm 1] and [BLS13, Theorem 1] for more details. Note in particular that the decomposition (1.2.9) depends on a choice of Gröbner basis for the Jacobian ideal J. This Gröbner basis should of course be fixed before comparing the reduced forms of A and A'.

An important remark about the decomposition (1.2.9) is that one should always carry out the reduction down to pole order 1, even though the minimal possible pole order

$$\alpha_0 = \min\{\alpha \geq 1 \mid \alpha d - n \geq 0\}$$

might be strictly larger than 1. This is not a contradiction: if $\alpha_0 > 1$ then we automatically have $r_\alpha = 0$ for all $1 \leq \alpha < \alpha_0$.

As a consequence of the reduced form (1.2.9) one obtains a method to explicitly write down a basis for the de Rham cohomology $H_{dR}^{n-1}(\mathbb{P}_K^{n-1} \setminus \mathcal{X}_K)$.

Proposition 1.2.4. *Write $R = K[x_1, \ldots, x_n]$. For $\alpha \in \{1, \ldots, n-1\}$, choose a set $\mathcal{M}_{\alpha d} \subset R$ consisting of monomials of degree $\alpha d - n$ whose classes modulo J form a basis of the K-vector space*

$$\left(\frac{R}{J}\right)_{\alpha d - n}.$$

Then the set of differential forms

$$\bigcup_{\alpha=1}^{n-1} \left\{ \frac{m\,\Omega}{\widetilde{\mathcal{G}}^{\alpha}} \mid m \in \mathcal{M}_{\alpha d} \right\} \tag{1.2.10}$$

is a basis for the de Rham cohomology space $H^{n-1}_{dR}(\mathbb{P}^{n-1}_K \setminus \mathcal{X}_K)$.

Proof. The idea is that a polynomial r_α appearing in the reduced form (1.2.9) is zero if and only if the pole order of

$$\frac{r_\alpha\,\Omega}{\widetilde{\mathcal{G}}^{\alpha}}$$

can be reduced, if and only if $r_\alpha \in J$. See [BLS13, Proposition 2] for a detailed proof. $\square$

A basis as in proposition 1.2.4 can be effectively computed. Indeed, for a fixed value α the set of all monomials of degree $\alpha d - n$ generates the space $(R/J)_{\alpha d-n}$. Any generating set can be thinned out to a basis, requiring only linear algebra in $(R/J)_{\alpha d-n}$. If one has a method to test whether an element of R belongs to J then this comes down to the same thing as doing linear algebra in $R_{\alpha d-n}$. So we see that everything can be made effective by calculating a Gröbner basis of J.

Proposition 1.2.4 also gives a method to write any differential form in terms of the basis (1.2.10). Indeed, after computing the reduced form (1.2.9) we may interpret the polynomials r_α as elements of $(R/J)_{\alpha d-n}$. Then it suffices to write each $r_\alpha \in (R/J)_{\alpha d-n}$ in terms of the basis $\mathcal{M}_{\alpha d}$. This only requires linear algebra techniques in $(R/J)_{\alpha d-n}$, which can be made effective.

It should be noted that the proof of [BLS13, Proposition 2] implicitly contains a construction for a *special* basis (1.2.10) such that the coordinates of a differential form can be read off *directly* from the Griffiths-Dwork reduction (1.2.9). We will come back to this point in chapter 4.

Approximation of the Frobenius matrix

We now give a short overview of the paper [AKR11].

Fix a finite field $k = \mathbb{F}_q$ and write $\mathcal{V} = W(k)$. As usual K denotes the fraction field of $\mathcal{V}$. Now consider a smooth hypersurface $X = Z(\widetilde{g}) \subset \mathbb{P}^{n-1}_k$ of degree d together with a smooth lift $\mathcal{X} = Z(\widetilde{\mathcal{G}}) \subset \mathbb{P}^{n-1}_{\mathcal{V}}$. The scheme $\mathbb{P}^{n-1}_{\mathcal{V}} \setminus \mathcal{X}$ can be written as $\operatorname{Spec} B$, where

$$B = \mathcal{V}[x_1, \ldots, x_n]_{(\widetilde{\mathcal{G}})} = \left\{ \frac{f(x_1, \ldots, x_n)}{\widetilde{\mathcal{G}}^t} \mid \deg f = t \cdot \deg \widetilde{\mathcal{G}} \right\}.$$

The first step towards understanding the Monsky-Washnitzer cohomology space $H^{n-1}_{MW}(\mathbb{P}^{n-1}_k \setminus X)$ is to write down an explicit Frobenius lift F on the weak completion $B^\dagger$. One such lift is given by the formal rules

$$F(x_i) = x_i^q \text{ for } i = 1, \dots, n$$

and

$$\begin{aligned} F(\widetilde{\mathcal{G}}^{-1}) &= F(\widetilde{\mathcal{G}})^{-1} \\ &= \left[\widetilde{\mathcal{G}}^q + \left(F(\widetilde{\mathcal{G}}) - \widetilde{\mathcal{G}}^q\right)\right]^{-1} \\ &= \sum_{i \geq 0} (-1)^i \frac{(F(\widetilde{\mathcal{G}}) - \widetilde{\mathcal{G}}^q)^i}{\widetilde{\mathcal{G}}^{q(i+1)}}. \end{aligned}$$

For a homogeneous polynomial A of degree $t \cdot d$ these formal rules give the equation

$$F\left(\frac{A}{\widetilde{\mathcal{G}}^t}\right) = \sum_{i \geq 0} \binom{t+i-1}{i} \frac{F(A) \cdot (\widetilde{\mathcal{G}}^q - F(\widetilde{\mathcal{G}}))^i}{\widetilde{\mathcal{G}}^{q(i+t)}}. \tag{1.2.11}$$

It is easy to see that the polynomial $\widetilde{\mathcal{G}}^q - F(\widetilde{\mathcal{G}})$ is divisible by p. Indeed, by reducing modulo p we obtain

$$\widetilde{g}(x_1, \dots, x_n)^q - \widetilde{g}(x_1^q, \dots, x_n^q) = 0.$$

Here we have also used the identity $a^q = a$ for every $a \in \mathbb{F}_q$. In this way we find that the i-th term of the sum (1.2.11) is divisible by p^i. Since the degree of the i-th term is a linear function of i, we see that the Frobenius lift is indeed an element of $B^\dagger$.

The Frobenius action on differential forms is completely determined by the rule

$$F(dx_i) = d(F(x_i)) = d(x_i^q) = q x_i^{q-1} dx_i.$$

From this we see that

$$F(\Omega) = q^{n-1} \cdot x_1^{q-1} \cdot \ldots \cdot x_n^{q-1} \cdot \Omega.$$

For the $(n-1)$-forms on the complement $\mathbb{P}_K^{n-1} \setminus \mathcal{X}_K$ we obtain the formula

$$F\left(\frac{A\,\Omega}{\widetilde{\mathcal{G}}^t}\right) = q^{n-1} \sum_{i \geq 0} \binom{t+i-1}{i} \frac{x_1^{q-1} \cdot \ldots \cdot x_n^{q-1} \cdot F(A) \cdot (\widetilde{\mathcal{G}}^q - F(\widetilde{\mathcal{G}}))^i\,\Omega}{\widetilde{\mathcal{G}}^{q(i+t)}} \tag{1.2.12}$$

where A is now of degree $td - n$.

In order to approximate the Frobenius matrix on $H_{MW}^{n-1}(\mathbb{P}_k^{n-1} \setminus X)$ one first writes down a basis for $H_{dR}^{n-1}(\mathbb{P}_K^{n-1} \setminus \mathcal{X}_K)$, using the method that was described in proposition 1.2.4. By the result from theorem 1.2.2 this can be thought of as a basis for $H_{MW}^{n-1}(\mathbb{P}_k^{n-1} \setminus X)$. The idea of the algorithm in [AKR11] is to *truncate* the images (1.2.12) of the basis elements. By truncating at a certain term N one obtains a set of differential $(n-1)$-forms that represent elements of $H^{n-1}_{dR}(\mathbb{P}^{n-1}_K \setminus \mathcal{X}_K)$. After computing the Griffiths-Dwork reductions of these differential forms one can write the corresponding de Rham cohomology classes as a linear combination of the basis. In other words: we obtained a recipe that calculates a certain matrix with entries in K.

The most important result in [AKR11] states that the resulting matrix is an approximation of the Frobenius matrix on $H^{n-1}_{MW}(\mathbb{P}^{n-1}_k \setminus X)$. It is moreover possible to control the precision of the result by carefully choosing the index N at which to truncate the sums (1.2.12). For any required precision on the end result, one can explicitly calculate a sufficiently large value of N.

The precise formulation of these results can be found in paragraphs 3.4 and 3.5 of [AKR11]. One important thing to note is that the cited results do *not* hold with respect to any choice of basis for $H^{n-1}_{dR}(\mathbb{P}^{n-1}_K \setminus \mathcal{X}_K)$. The basis should be constructed as described in [AKR11, Definition 3.4.2].

1.2.4 Rigid cohomology

As we mentioned before, rigid cohomology is a generalization of Monsky-Washnitzer cohomology that works for general schemes. There are essentially two (mostly equivalent) ways of defining rigid cohomology. We will use the most classical definition that is due to Berthelot [Ber96] [Ber97b]. We will use the book [LS07] as our reference for this formulation of rigid cohomology. There is a newer and conceptually cleaner definition based on topos theory [LS11]. We will briefly discuss this alternative definition in paragraph 1.2.5.

The classical definition of rigid cohomology is built around the notion of a *frame*.

Definition 1.2.5. A *frame* is a series of immersions $(X \subset Y \subset P)$ where X and Y are schemes over k and P is a formal scheme over $\mathcal{V}$. The immersion $X \subset Y$ is required to be open and $Y \subset P$ is assumed to be a closed immersion of Y into the closed fiber of P. A morphism from a frame $(X' \subset Y' \subset P')$ to a frame $(X \subset Y \subset P)$ consists of three morphisms $X' \to X$, $Y' \to Y$ and $u\colon P' \to P$ that respect all the given inclusions. In other words: the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd} \tag{1.2.13}$$

is commutative. We let S denote the frame $(\operatorname{Spec} k \subset \operatorname{Spec} k \subset \operatorname{Spf} \mathcal{V})$. In this way, any frame $(X \subset Y \subset P)$ is a frame over S. Every morphism of frames will be assumed to be an S-morphism. A scheme X is called *realizable* if there exists a frame $(X \subset Y \subset P)$ with Y proper and P smooth in a neighbourhood of X. It is easy to show that for any morphism of realizable schemes $f\colon X' \to X$ there exists a morphism of frames

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd} \tag{1.2.14}$$

such that $(X' \subset Y' \subset P')$ resp. $(X \subset Y \subset P)$ is a realization of X' resp. of X and such that u is smooth in a neighbourhood of X'. See section 8.2 in [LS07]. Such a morphism of frames is called a *realization* of f.

We will assume from now on that every k-scheme that we consider is realizable. This is not much of a restriction as quasi-projective schemes are obviously realizable. Indeed, if X is locally closed in $\mathbb{P}^n_k$ then we may take $Y = \overline{X}$ the closure of X in $\mathbb{P}^n_k$ and $P = \widehat{\mathbb{P}^n_{\mathcal{V}}}$.

Recall from paragraph 1.2.1 that we assume the formal scheme P in the definition above to be separated and topologically of finite type over $\mathcal{V}$. For such a P we may consider the generic fiber P_K, which is a quasi-compact separated rigid analytic space over K. See [Ray74] or [Nic08] for details. From this construction we also obtain a *specialization map* $\mathrm{sp}\colon P_K \to P$. The underlying topological space of P is the same as the topology of the closed fiber P_k. Therefore we can also interpret the specialization map as a continuous map

$$P_K \to P_k. \tag{1.2.15}$$

From this set-up one can define several new objects. We refer to [LS07] for the precise definitions.

- Given a frame $(X \subset Y \subset P)$ one can define the *tubes* $]X[_P$ and $]Y[_P$, which are defined as the inverse image of X resp. Y under the map (1.2.15). As such they are rigid analytic spaces over K.

- With the same notations as in the previous point, let $\mathcal{O}_{]Y[_P}$ denote the structure sheaf of the tube $]Y[_P$. Then one can define another sheaf $j_X^\dagger \mathcal{O}_{]Y[_P}$. This is called the sheaf of *overconvergent functions* on $]Y[_P$ w.r.t. the inclusion $X \subset Y$. See chapter 5 in [LS07] for more details.

- Given a morphism of frames as in (1.2.13), with the rightmost arrow denoted by $u\colon P' \to P$, there is a canonical map $u_K\colon P'_K \to P_K$ on the associated rigid analytic spaces. Moreover, we can consider the restriction $u_K\colon]Y'[_{P'} \to]Y[_P$ on the tubes. If E' resp. E denotes an $\mathcal{O}_{]Y'[_{P'}}$-module resp. an $\mathcal{O}_{]Y[_P}$-module then we can consider the sheaves $u_{K*}E'$ and $u_K^* E$.

With these definitions in hand we are now ready to state the definition of rigid cohomology with constant coefficients.

Definition 1.2.6. Consider a k-scheme X with structural morphism $p\colon X \to \operatorname{Spec} k$. Also choose a realization $(X \subset Y \subset P)$, which comes with a structural morphism to S. In particular we have an arrow $p_K\colon P_K \to \mathrm{Sp}(K)$ on the generic fibers. Then the *rigid cohomology with constant coefficients* of X is defined as the derived pushforward

$$(\mathbb{R}p_{K*})\, j_X^\dagger \Omega^\bullet_{]Y[_P/K} \tag{1.2.16}$$

where $\Omega^\bullet_{]Y[_P/K}$ denotes the canonical complex of differential forms on $]Y[_P$. The derived pushforward must be seen as an object of $D^+(\mathcal{O}_{\mathrm{Sp}(K)}\text{-Mod})$, the bounded below derived category of $\mathcal{O}_{\mathrm{Sp}(K)}$-modules. In more concrete terms, (1.2.16) may be interpreted as a complex of K-vector spaces. The i-th cohomology of the complex (1.2.16) is a K-vector space that is also denoted by

$$H^i_{rig}(X) \coloneqq \left(\mathbb{R}^i p_{K*}\right) j_X^\dagger \Omega^\bullet_{]Y[_P/K}.$$

There are a number of remarks to make about this definition.

- In the definition above we have *chosen* a realization, so one should make sure that the definition is independent of this choice. This is in fact a deep theorem. See [LS07, Proposition 6.5.3].

- The motivation behind the definition is as follows. Since X is a (possibly) singular scheme over a field of characteristic $p > 0$, defining a de Rham-type cohomology theory in terms of differentials on X itself is out of the question. The idea to solve this issue is similar to what happens in the definition of Monsky-Washnitzer cohomology. We lift the situation to $]Y[_P$, which is a *smooth* structure over the field K. Just like Monsky-Washnitzer cohomology, rigid cohomology is defined as an *overconvergent* de Rham-type cohomology on the lifted structure. The overconvergence is needed to ensure that the resulting cohomology theory is well-defined, although for rigid cohomology this is much harder to prove then for Monsky-Washnitzer cohomology.

- As we mentioned before, if X is a smooth affine scheme then there is a canonical isomorphism $H^i_{rig}(X) \cong H^i_{MW}(X)$ for all $i \geq 0$ [Ber97b, Proposition 1.10].

- It is known that the cohomology spaces $H^i_{rig}(X)$ of a k-variety X are finite-dimensional over K. For smooth X this is shown in [Ber97b, Théorème 3.1]. The proof for the general case relies on cohomological descent, which we will briefly discuss in paragraph 1.2.5.

- The definition above doesn't specify how the Frobenius map on rigid cohomology is defined. We will elaborate on this in chapter 2.

Rigid cohomology with supports

As we mentioned in paragraph 1.1.2, we would like to study the local cohomology $H^\bullet_{rig,\{x\}}(X)$ of a closed point $x \in X$. For this reason we should also mention the definition of *rigid cohomology with supports in a closed subset*.

Consider a scheme X together with a closed subscheme $Z \subset X$. Also choose a realization $(X \subset Y \subset P)$ as before. With this data one can construct a functor $\underline{\Gamma}^\dagger_Z$ on the category of $j^\dagger_X \mathcal{O}_{]Y[_P}$-modules. This functor is called the *functor of overconvergent sections with support in* $]Z[_P$. The precise definition can be found in section 5.2 of [LS07]. However, the most important property to remember about $\underline{\Gamma}^\dagger_Z$ is that it is characterized by a short exact sequence

$$0 \longrightarrow \underline{\Gamma}^\dagger_Z E \longrightarrow E \longrightarrow j^\dagger_{X\setminus Z} E \longrightarrow 0 \tag{1.2.17}$$

for any $j^\dagger_X \mathcal{O}_{]Y[_P}$-module E.

Definition 1.2.7. Assume the same notations as in definition 1.2.6 and let $Z \subset X$ be a closed subscheme. The rigid cohomology with constant coefficients of X with *supports in* Z is defined as

$$(\mathbb{R}p_{K*})\, \underline{\Gamma}^\dagger_Z j^\dagger_X \Omega^\bullet_{]Y[_P/K},$$

which again must be seen as an object of the derived category $D^+(\mathcal{O}_{\mathrm{Sp}(K)}\text{-Mod})$. In this case we use the notation

$$H^i_{rig,Z}(X) := \left(\mathbb{R}^i p_{K*}\right) \underline{\Gamma}^\dagger_Z j^\dagger_X \Omega^\bullet_{]Y[_P/K}.$$

An important fact to remember about cohomology with supports is that there is a *long exact sequence of rigid cohomology with supports*:

$$\ldots \longrightarrow H^i_{rig,Z}(X) \longrightarrow H^i_{rig}(X) \longrightarrow H^i_{rig}(X \setminus Z) \longrightarrow H^{i+1}_{rig,Z}(X) \longrightarrow \ldots \tag{1.2.18}$$

The existence of this sequence can easily be derived from the short exact sequence (1.2.17).

The Gysin isomorphism and the Künneth formula

We now recall two fundamental results about the rigid cohomology of *smooth* schemes.

Theorem 1.2.8. *Consider a smooth scheme X and a smooth closed subscheme Z of pure codimension c. For every $i \geq 0$ there is a canonical*

Frobenius-equivariant isomorphism

$$H^i_{rig,Z}(X) \xrightarrow{\sim} H^{i-2c}_{rig}(Z)(-c). \tag{1.2.19}$$

Proof. See [Ber97a] and [CLS99, Corollaire 2.1.3]. □

The isomorphism (1.2.19) is called the *Gysin isomorphism.* If $i - 2c < 0$ then the right-hand side should be read as the zero space. The extra brackets $(-c)$ signal a *Frobenius twist.* If we take the q-power Frobenius then this means that we consider the K-space $H^{i-2c}_{rig}(Z)$ together with the σ-linear map that is given by q^c times the Frobenius map $\mathrm{Fr}^{(q)}$. The term "Frobenius-equivariant" indicates that the isomorphism (1.2.19) is compatible with this *modified* σ-linear map.

The Gysin isomorphism transforms the long exact sequence (1.2.18) into the *Gysin sequence*

$$\ldots \to H^i_{rig}(X \setminus Z) \to H^{i+1-2c}_{rig}(Z)(-c) \to H^{i+1}_{rig}(X) \to H^{i+1}_{rig}(X \setminus Z) \to \ldots \tag{1.2.20}$$

Another strong theorem about the rigid cohomology of smooth schemes is the *Künneth formula.*

Theorem 1.2.9. *Let X_1 and X_2 be two smooth schemes. Then for all $l \geq 0$ there is a canonical Frobenius-equivariant isomorphism*

$$H^l_{rig}(X_1 \times X_2) \xrightarrow{\sim} \bigoplus_{i+j=l} H^i_{rig}(X_1) \otimes_K H^j_{rig}(X_2). \tag{1.2.21}$$

Proof. See [Ber97a]. The Frobenius-equivariance of the Künneth formula is discussed in [Tsu99]. □

Rigid cohomology with compact supports

We end this paragraph by saying a few words about *rigid cohomology with compact supports*, which is a variation of rigid cohomology. We will not go into many details, as it suffices to know that for *smooth* schemes the two theories are related to each other by the *Poincaré duality theorem.* For a smooth scheme X of pure dimension d with $Z \subset X$ a closed subscheme, Poincaré duality gives an isomorphism

$$H^i_{rig,c}(Z) \xrightarrow{\sim} H^{2d-i}_{rig,Z}(X)(d)^\vee$$

for any $i \geq 0$. The cohomology space on the left is the one with compact supports (signalled by the subscript c), the space on the right is the one from definition 1.2.7. A more complete statement with proof can be found in [Ber97a]. Also see [CLS99] for details about the Frobenius twist.

The Gysin isomorphism (1.2.19) can be seen as a direct corollary of Poincaré duality. Also, the Gysin sequence (1.2.20) can be obtained by Poincaré duality from the *long exact sequence of rigid cohomology with compact supports*:

$$\ldots \to H^i_{rig,c}(X \setminus Z) \to H^i_{rig,c}(X) \to H^i_{rig,c}(Z) \to H^{i+1}_{rig,c}(X \setminus Z) \to \ldots \tag{1.2.22}$$

In this text we prefer to avoid rigid cohomology with compact supports whenever possible. Or rather: we apply Poincaré duality as early as possible.

1.2.5 The overconvergent site and cohomological descent

To end this introductory part on p-adic cohomology we would like to say a few words about the *overconvergent site* (or *overconvergent topos*). The main reference for this topic is [LS11].

For any k-scheme X there is a site $\mathrm{AN}^\dagger(X)$ called the *overconvergent analytic site over* X. The associated topos is denoted by $X_{\mathrm{AN}^\dagger}$. Every morphism of k-schemes $f\colon X \to Y$ induces a morphism of topoi

$$(f^{-1}_{\mathrm{AN}^\dagger}, f_{\mathrm{AN}^\dagger *})\colon X_{\mathrm{AN}^\dagger} \to Y_{\mathrm{AN}^\dagger}.$$

These functors can also be written as f^{-1} and f_* if there is no danger of confusion.

This construction allows one to define rigid cohomology inside of the general framework of topos theory. This alternative definition removes the need to choose a realization. The topos-theoretic definition of rigid cohomology is based on the following result.

Proposition 1.2.10. *Let X be a k-scheme with structural morphism $p\colon X \to$ $\operatorname{Spec} k$. Then there is an equivalence of categories between the overconvergent isocrystals on X (also see paragraph 2.1.2) and the finitely presented $\mathcal{O}^\dagger_X$-modules (which are objects of $X_{AN^\dagger}$). Under this equivalence the constant isocrystal $\mathcal{O}_{X/K}$ corresponds to the module $\mathcal{O}^\dagger_X$. The objects on both sides of the equivalence are also cohomologically compatible. In the case of constant coefficients there is an isomorphism*

$$H^i_{rig}(X) \xrightarrow{\sim} \mathbb{R}^i p_{AN^\dagger *}\mathcal{O}^\dagger_X$$

for every $i \geq 0$.

Proof. See [LS11, Theorem 4.6.7] and [LS11, Corollary 4.6.8]. □

The reason why we present the topos-theoretic definition of rigid cohomology is that it has recently been shown that the topos-theoretic setting offers a natural environment for the definition of *cohomological descent* [ZB14]. Cohomological descent (in the context of rigid cohomology) is a technique that allows one to generalize certain properties of the rigid cohomology of smooth schemes. For example, some of the early results about rigid cohomology state that if X is a *smooth* k-scheme then $H^i_{rig}(X)$ is finite-dimensional [Ber97b, Théorème 3.1] and the Frobenius action is invertible [CLS99, Proposition 2.1.4]. The technique of cohomological descent allows one to generalize these results to singular k-schemes. See for example [Tsu03, Theorem 5.1.1].

Several authors have already obtained strong results about cohomological descent for rigid cohomology [Tsu03] [CT03]. However, the recent paper [ZB14] puts all the previous results in a more natural setting. It heavily relies on the topos-theoretic definition of rigid cohomology. We will only use cohomological descent in section 3.2. For this reason we only present the aspects of cohomological descent that we really need. In particular we avoid most of the abstract (but powerful) machinery from [ZB14].

The theory of cohomological descent is built around the notion of a *simplicial object*. An *augmented simplicial object* $a\colon X_\bullet \to Y$ over a scheme Y consists of a collection of schemes $\{X_n\}_{n\geq 0}$ together with a collection of morphisms $\{a_n\colon X_n \to Y\}_{n\geq 0}$ as well as a set of morphisms $\{a_n^j\colon X_n \to X_{n-1}\}_{j=0}^n$ for every $n \geq 1$. These morphism are required to satisfy the conditions

$$a_n = a_0 \circ a_1^{j_1} \circ \ldots \circ a_n^{j_n}$$

for every $n \geq 1$ and for every choice of indices $j_1, \ldots, j_n$. This is not the complete definition of a simplicial object, but it is detailed enough for our needs. See the notes [Con03] for a good overview of simplicial objects.

There is a special class of simplicial objects called $\mathbf{P}-hypercovers$, where $\mathbf{P}$ is a property of schemes that is invariant under base extensions. We will not repeat the definition of a $\mathbf{P}$-hypercover, since for our applications we will always obtain a proper hypercover from the alteration theorem of de Jong. There is no need to construct a proper hypercover "by hand" and therefore it is enough to remember that a $\mathbf{P}$-hypercover of a scheme Y is just a special kind of simplicial object $a\colon X_\bullet \to Y$.

Let $a\colon X_\bullet \to Y$ be an augmented simplicial object and consider an $\mathcal{O}^\dagger_Y$-module $\mathcal{F} \in Y_{AN^\dagger}$. The module $a^*\mathcal{F}$ is given by the collection of ordinary pullbacks $a_n^*\mathcal{F} \in (X_n)_{AN^\dagger}$. More generally, a module $\mathcal{G}$ on $X_\bullet$ is given by a collection of $\mathcal{O}^\dagger_{X_n}$-modules $\mathcal{G}_n \in (X_n)_{AN^\dagger}$, together with compatible morphisms $(a^j_{n+1})^*\mathcal{G}_n \to \mathcal{G}_{n+1}$ [2]. The derived pushforward $\mathbb{R}a_*\mathcal{G}$ is defined as the equalizer of the following morphisms in $D^+(\mathcal{O}^\dagger_Y\text{-Mod})$:

$$\mathbb{R}(a_0)_*\mathcal{G}_0 \rightrightarrows \mathbb{R}(a_1)_*\mathcal{G}_1 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \ldots$$

where the arrows $\mathbb{R}(a_n)_*\mathcal{G}_n \to \mathbb{R}(a_{n+1})_*\mathcal{G}_{n+1}$ are given by applying $\mathbb{R}(a_n)_*$ to

[2] The formulation is intentionally vague. The objects $a^*\mathcal{F}$ and $\mathcal{G}$ should be seen as modules on the *simplicial site* $X_\bullet$. See [ZB14] for details.

the composition

$$\mathcal{G}_n \to \mathbb{R}(a^j_{n+1})_*(a^j_{n+1})^*\mathcal{G}_n \to \mathbb{R}(a^j_{n+1})_*\mathcal{G}_{n+1}.$$

Note that if we take $\mathcal{G} = a^*\mathcal{F}$ then there is a canonical map $\mathcal{F} \to \mathbb{R}a_*a^*\mathcal{F}$. Indeed, since there are compatible canonical maps $\mathcal{F} \to \mathbb{R}(a_n)_*(a_n)^*\mathcal{F}$ for every n, we get a canonical map to the equalizer. The augmented simplicial object a is said to be *of cohomological descent* w.r.t. $\mathcal{F}$ if the map $\mathcal{F} \to \mathbb{R}a_*a^*\mathcal{F}$ is an isomorphism.

In chapter 3 we will make use of the following property:

Proposition 1.2.11. *A proper hypercover $a\colon X_\bullet \to Y$ is of cohomological descent with respect to finitely presented $\mathcal{O}^\dagger_Y$-modules $\mathcal{F} \in Y_{AN^\dagger}$.*

Proof. See [ZB14, Theorem 1.1]. □

This finishes our overview of p-adic cohomology. In the rest of this thesis we will sometimes use standard notations and terminology related to rigid cohomology without explicitly defining them. The relevant definitions can all be found in [LS07]. Note that chapter 2 contains some more introductory material about rigid cohomology, which will only be needed in that chapter.

Chapter 2

Invariance of local rigid cohomology

2.1 Introduction and statement of results

Let $x \in X$ be a singular closed point on a k-scheme. Then we may consider the local rigid cohomology $H^\bullet_{rig,\{x\}}(X)$ at this singular point. The goal of this chapter is to prove that the local rigid cohomology is an invariant of the singularity $x \in X$. Indeed, as a direct consequence of this chapter's main theorem 2.1.8 we will prove the following property.

Theorem 2.1.1. *Let $x' \in X'$ and $x \in X$ be two closed points on k-schemes. Assume that these two points are contact equivalent. Then for all i there exists an isomorphism*

$$H^i_{rig,\{x\}}(X) \xrightarrow{\sim} H^i_{rig,\{x'\}}\left(X'\right)$$

on the local rigid cohomology. This isomorphism is moreover compatible with the Frobenius action on rigid cohomology.

This implies in particular that the local rigid cohomology is only interesting at singular points.

The rest of this chapter's introduction is organized as follows. First we recall the notion of contact equivalence for schemes over arbitrary fields. We also recall an important reformulation of this definition that is due to Artin. Then we expand a bit on the notion of rigid cohomology with nonconstant coefficients. In paragraph 2.1.3 we recall how the functoriality of rigid cohomology and its canonical Frobenius action are defined in terms of base change maps. After this we formulate our main theorem 2.1.8, which is a more refined version of theorem 2.1.1.

The proof of theorem 2.1.8 will then be covered in section 2.2.

2.1.1 Equivalence of singularities

Consider two points on k-schemes $x' \in X'$ and $x \in X$. We say that these two points are *contact equivalent* if there exists an isomorphism $\widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X',x'}$ on the completed local rings. We denote this by $(X', x') \sim_c (X, x)$. This definition of contact equivalence is valid over any base field. It is a well-known fact that over $\mathbb{C}$ one recovers the classical analytic definition. See [Art68, Corollary 1.6].

In this chapter we will need to use the following reformulation of contact equivalence. It is due to Artin.

Proposition 2.1.2. *Two points $x' \in X'$ and $x \in X$ are contact equivalent if and only if there exists another scheme X'' together with a point $x'' \in X''$ and two morphisms $f' \colon X'' \to X'$ and $f \colon X'' \to X$ such that:*

i) $f'(x'') = x'$ and f' induces an isomorphism $k(x') \xrightarrow{\sim} k(x'')$ on the residue fields.

ii) $f(x'') = x$ and f induces an isomorphism $k(x) \xrightarrow{\sim} k(x'')$ on the residue fields.

iii) f' and f are étale at x''.

Proof. For the "if" part of the proposition it suffices to observe that conditions i) through iii) imply that the Henselizations $\mathcal{O}^h_{X,x}$ and $\mathcal{O}^h_{X',x'}$ are isomorphic. Indeed, another way to formulate the conditions of the proposition is that X'' is a neighbourhood of x in the étale site $X_{ét}$ as well as a neighbourhood of x' in the étale site $X'_{ét}$. Hence the local rings of x and x' w.r.t. the étale site are isomorphic. The isomorphism $\mathcal{O}^h_{X,x} \cong \mathcal{O}^h_{X',x'}$ implies that the completions of the local rings are isomorphic as well.

The "only if" part is proved in [Art69, Corollary 2.6]. □

Note that if $x \in X$ is a smooth point then there exists an étale morphism $f \colon U \to \mathbb{A}^d_k$ where $U \subset X$ is a neighbourhood of x and $d = \dim_x X$ is the dimension of X at x. From this one can easily show that two smooth closed points $x \in X$ and $x' \in X'$ are contact equivalent if and only if $\dim_x X = \dim_{x'} X'$ and $k(x) \cong k(x')$. See for example [Liu02, Exercise 6.2.1]. Combining this observation with theorem 2.1.1 we see that for a smooth closed point $x \in X$ we have $H^i_{rig,\{x\}}(X) = 0$ for every $i > 0$. In other words: the local rigid cohomology is only interesting for singular points.

In the statement of our main theorem we will use a reformulation of the étale characterization of contact equivalence. For this we make the following definition.

Definition 2.1.3. Consider two points on k-schemes $x' \in X'$ and $x \in X$. We will write $(X', x') \succ (X, x)$ if there exists an open neighbourhood $U_{x'}$ of x' and a morphism $f \colon U_{x'} \to X$ such that:

i) $f(x') = x$ and f induces an isomorphism $k(x) \xrightarrow{\sim} k(x')$ on the residue fields.

ii) f is étale at x'.

iii) $f^{-1}(x) = \{x'\}$.

Proposition 2.1.2 now gives us an obvious connection between the notion of contact equivalence and the conditions in definition 2.1.3.

Proposition 2.1.4. *Assume that $x' \in X'$ and $x \in X$ are closed points. Then we have that $(X', x') \sim_c (X, x)$ if and only if there exists a pair (X'', x'') such that $(X'', x'') \succ (X', x')$ and $(X'', x'') \succ (X, x)$.*

Proof. First assume that there exists a pair (X'', x'') such that $(X'', x'') \succ (X', x')$ and $(X'', x'') \succ (X, x)$. Then there exist open neighbourhoods $U'_{x''}$ and $U_{x''}$ of x'' in X'' together with maps $f' \colon U'_{x''} \to X'$ and $f \colon U_{x''} \to X$ satisfying the conditions of definition 2.1.3. Then the maps f' and f restricted to $U'_{x''} \cap U_{x''}$ satisfy the conditions from proposition 2.1.2. Conversely, assume that $(X', x') \sim_c (X, x)$. Then we fix morphisms $f' \colon X'' \to X' \colon x'' \mapsto x'$ and $f \colon X'' \to X \colon x'' \mapsto x$ that satisfy the conditions from proposition 2.1.2. After replacing X'' by an open neighbourhood of x'' we may assume that f' and f are étale. It follows that the fibers $(f')^{-1}(x')$ and $f^{-1}(x)$ consist of a finite union of closed points. We may therefore shrink X'' and assume that $(f')^{-1}(x') = \{x''\}$ and $f^{-1}(x) = \{x''\}$. Then we have $(X'', x'') \succ (X', x')$ and $(X'', x'') \succ (X, x)$. $\square$

2.1.2 Overconvergent isocrystals

In section 2.2 we will prove a more general version of theorem 2.1.1, which is formulated for rigid cohomology with nonconstant coefficients. The reason for this formulation is that the general case is not substantially more difficult to prove than the special case with constant coefficients.

In this paragraph we briefly recall the notion of an *overconvergent isocrystal*. These objects serve as the category of coefficients for rigid cohomology.

Definition 2.1.5. Let X be a scheme with a realization $(X \subset Y \subset P)$. A *finitely presented overconvergent isocrystal* $\mathcal{F}$ on this frame consists of the following data:

i) A coherent $j^\dagger_{X'}\mathcal{O}_{]Y'[_{P'}}$-module $E_{P'}$ for every morphism of frames

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & P' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & P \end{array}$$

ii) An isomorphism

$$\varphi_u\colon u^\dagger E_{P'} \xrightarrow{\sim} E_{P''} \tag{2.1.1}$$

for every morphism of frames

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & P'' \arrow[d, "u"] \\
X' \arrow[r] & Y' \arrow[r] & P'
\end{tikzcd}$$

over $(X \subset Y \subset P)$. The functor $u^\dagger$ in equation (2.1.1) is called the *overconvergent pullback functor*. It is defined as $u^\dagger := j^\dagger_{X''} u^*_K$. The isomorphisms φ_u are also required to satisfy a cocycle condition. See chapter 7 of [LS07] for the complete definition.

The module $E_{P'}$ is called the *realization* of $\mathcal{F}$ on the frame $(X' \subset Y' \subset P')$.

It is proved in [LS07, Theorem 7.1.8] that if $(X \subset Y_1 \subset P_1)$ is another realization of X then there is an equivalence of categories between the overconvergent isocrystals on $(X \subset Y \subset P)$ resp. on $(X \subset Y_1 \subset P_1)$. Therefore one simply speaks about the *overconvergent isocrystals on X*. This category is denoted by $\mathrm{Isoc}^\dagger(X/S)$.

Now let $\mathcal{F}$ be an overconvergent isocrystal on X and $f\colon X' \to X$ a morphism of k-schemes with a realization as in the diagram below.

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\
X \arrow[r] & Y \arrow[r] & P
\end{tikzcd}$$

Then one defines the pullback $f^*\mathcal{F}$ to be the overconvergent isocrystal on X' whose realization on a frame $(X'' \subset Y'' \subset P'')$ over $(X' \subset Y' \subset P')$ is given by $E_{P''}$, the realization of $\mathcal{F}$ on $(X'' \subset Y'' \subset P'')$.

As an example, the constant overconvergent isocrystal on X, which is denoted by $\mathcal{O}_{X/K}$, is defined by the formula $E_{P'} = j^\dagger_{X'}\mathcal{O}_{]Y'[_{P'}}$. For a morphism $f\colon X' \to X$ we obviously have $f^*\mathcal{O}_{X/K} = \mathcal{O}_{X'/K}$.

If we fix a Frobenius map $x \mapsto x^{p^r}$ on k then we may also consider overconvergent isocrystals with an additional Frobenius structure. Indeed, take X a k-scheme with absolute Frobenius F_X and let $\mathcal{F}$ be an overconvergent isocrystal on X. Then a *Frobenius structure* on $\mathcal{F}$ is an isomorphism $\Phi\colon F_X^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. Note that F_X is in general not a morphism of k-schemes, so the definition of the pullback functor needs to be modified according to [LS07, Definition 8.3.1]. An overconvergent isocrystal $\mathcal{F}$ with a Frobenius structure is also called an *overconvergent F-isocrystal.* The overconvergent F-isocrystals on X form a subcategory of $\mathrm{Isoc}^\dagger(X/S)$, which is denoted by $\text{F-Isoc}^\dagger(X/S)$.

The constant overconvergent isocrystal $\mathcal{O}_{X/K}$ satisfies $F_X^*\mathcal{O}_{X/K} = \mathcal{O}_{X/K}$, therefore it has a natural Frobenius structure $\Phi = \mathrm{Id}$.

A consequence of the compatibility condition (2.1.1) is that every realization $E = E_P$ of an overconvergent isocrystal $\mathcal{F}$ on a frame $(X \subset Y \subset P)$ is equipped with an overconvergent stratification [LS07, Proposition 7.2.2]. According to [LS07, Proposition 7.2.13] this amounts to saying that E is a $j_X^\dagger \mathcal{O}_{]Y[_P}$-module with an overconvergent integrable connection over K. From this one can define the *de Rham complex* $E \otimes_{]Y[_P} \Omega^\bullet_{]Y[_P/K}$. We refer to paragraph 4.1.4 in [LS07] for more details. But note that for $\mathcal{F} = \mathcal{O}_{X/K}$ with realization $E = j_X^\dagger \mathcal{O}_{]Y[_P}$ the de Rham complex reduces to the complex $\Omega^\bullet_{]Y[_P/K}$ of canonical differential sheaves on $]Y[_P$.

With this new terminology we can state the definition of rigid cohomology with nonconstant coefficients.

Definition 2.1.6. Let $p\colon X \to \operatorname{Spec} k$ be a k-scheme with a closed subscheme $C \subset X$. Also consider a finitely presented overconvergent isocrystal $\mathcal{F} \in \mathrm{Isoc}^\dagger(X/S)$. Choose a realization $(X \subset Y \subset P)$ and let E be the realization of $\mathcal{F}$ on this frame. Also let $p_K\colon P_K \to \mathrm{Sp}(K)$ denote the map on generic fibers. The *rigid cohomology of X with coefficients in $\mathcal{F}$ and with supports in C* is defined as

$$(\mathbb{R}p_{K*})\, \underline{\Gamma}_C^\dagger E \otimes_{]Y[_P} \Omega^\bullet_{]Y[_P/K}. \tag{2.1.2}$$

The complex (2.1.2) can also be denoted as $\mathbb{R}p_{C,rig}\mathcal{F}$ if one wishes to hide the choice of realization and instead put emphasis on the object $\mathcal{F}$. For the cohomology of the complex (2.1.2) we use the notation

$$H^i_{rig,C}(X, \mathcal{F}) \coloneqq \left(\mathbb{R}^i p_{K*}\right) \underline{\Gamma}_C^\dagger E \otimes_{]Y[_P} \Omega^\bullet_{]Y[_P/K}.$$

The result [LS07, Proposition 6.5.3] again guarantees that this definition is independent of the choice of realization. If we take $\mathcal{F} = \mathcal{O}_{X/K}$ then we obviously recover the previous definition 1.2.7.

2.1.3 Base change maps, functoriality and Frobenius

Before we state our main theorem 2.1.8 we will review the *base change maps* that are needed to understand how rigid cohomology behaves w.r.t. morphisms of schemes. At the same time we introduce some convenient notation for section 2.2. In this sense our presentation is slightly different than in [LS07]. At the end of this paragraph we also recall the definition and basic properties of the Frobenius action on the rigid cohomology of an F-isocrystal.

Consider a commutative diagram of rigid analytic spaces

$$\begin{tikzcd}
V' \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & W' \arrow[d, "\beta"] \\
V \arrow[r, "\alpha"] & W
\end{tikzcd}$$

where β and β' are flat. Let E be an $\mathcal{O}_V$-module. Then there is a canonical *base change map*

$$\beta^* (\mathbb{R}\alpha_*) E \longrightarrow \left(\mathbb{R}\alpha'_*\right) (\beta')^* E. \tag{2.1.3}$$

For the construction of this base change map we refer to [Sta15, Tag 02N6] or to paragraph XVII.2 of [SGA4]. Also note that by the flatness assumption on β and β' we do not need to consider the left derived functors $\mathbb{L}\beta^*$ and $\mathbb{L}(\beta')^*$.

The map (2.1.3) is the starting point for the definition of the base change map of rigid cohomology. Let

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\
X \arrow[r] & Y \arrow[r] & P
\end{tikzcd} \tag{2.1.4}$$

be a flat morphism of frames. Also choose two closed subschemes $C' \subset X'$ and $C \subset X$ such that $f^{-1}(C) \subset C'$. Let E be a $j^\dagger_X \mathcal{O}_{]Y[_P}$-module with an integrable connection over K.

Since (2.1.4) is flat we know by [LS07, Corollary 3.3.6] that there exists a strict neighbourhood V' of $]X'[_{P'}$ in $]Y'[_{P'}$ such that the morphism $u_K \colon]Y'[_{P'} \to]Y[_P$ is flat on V'. It follows from [LS07, Proposition 6.2.2] that restricting to V' has no effect on cohomology, therefore we may work as if u_K were flat. We can then apply the base change map (2.1.3) coming from the diagram

$$\begin{tikzcd}
]Y'[_{P'} \arrow[r, "u_K"] \arrow[d, "u_K"] &]Y[_P \arrow[d, "\mathrm{Id}"] \\
]Y[_P \arrow[r, "\mathrm{Id}"] &]Y[_P
\end{tikzcd}$$

to the de Rham complex $\underline{\Gamma}^\dagger_C E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K}$. In this way we obtain a morphism

$$(u\star)_1 \colon \underline{\Gamma}^\dagger_C E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K} \longrightarrow (\mathbb{R}u_{K*})\, u_K^* \left(\underline{\Gamma}^\dagger_C E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K}\right).$$

By applying the morphism of functors $\mathrm{Id} \to j^\dagger_{X'}$ to the pullback of the de

Rham complex we obtain another morphism

$$(u\star)_2\colon\ (\mathbb{R}u_{K*})\, u_K^*\left(\underline{\Gamma}_C^\dagger E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K}\right) \longrightarrow (\mathbb{R}u_{K*})\, u^\dagger \left(\underline{\Gamma}_C^\dagger E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K}\right).$$

We also have a map

$$u_K^* \left(\underline{\Gamma}_C^\dagger E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K}\right) \longrightarrow u_K^* \underline{\Gamma}_C^\dagger E \otimes_{\mathcal{O}_{]Y'[_{P'}}} \Omega^\bullet_{]Y'[_{P'}/K} \tag{2.1.5}$$

from the pullback of the de Rham complex to the de Rham complex of the pullback. This map can easily be constructed by combining the canonical arrow $u_K^*\Omega^1_{]Y[_P} \to \Omega^1_{]Y'[_{P'}}$ with the definition of the de Rham complex. See for example to introduction of [KO68]. From (2.1.5) we find another map

$$(u\star)_3\colon\ (\mathbb{R}u_{K*})\, u^\dagger\left(\underline{\Gamma}_C^\dagger E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K}\right) \longrightarrow (\mathbb{R}u_{K*})\, u^\dagger \underline{\Gamma}_C^\dagger E \otimes_{\mathcal{O}_{]Y'[_{P'}}} \Omega^\bullet_{]Y'[_{P'}/K}.$$

Finally we have a canonical map $u^\dagger \underline{\Gamma}_C^\dagger E \longrightarrow \underline{\Gamma}_{C'}^\dagger u^\dagger E$, which we will study more in paragraph 2.2.1. From this map we obtain a morphism

$$(u\star)_4\colon\ (\mathbb{R}u_{K*})\, u^\dagger \underline{\Gamma}_C^\dagger E \otimes_{\mathcal{O}_{]Y'[_{P'}}} \Omega^\bullet_{]Y'[_{P'}/K} \longrightarrow (\mathbb{R}u_{K*})\, \underline{\Gamma}_{C'}^\dagger u^\dagger E \otimes_{\mathcal{O}_{]Y'[_{P'}}} \Omega^\bullet_{]Y'[_{P'}/K}.$$

Definition 2.1.7. The canonical map

$$u\star\colon \underline{\Gamma}_C^\dagger E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K} \longrightarrow (\mathbb{R}u_{K*})\, \underline{\Gamma}_{C'}^\dagger u^\dagger E \otimes_{\mathcal{O}_{]Y'[_{P'}}} \Omega^\bullet_{]Y'[_{P'}/K} \tag{2.1.6}$$

that is given by the composition

$$u\star = (u\star)_4 \circ (u\star)_3 \circ (u\star)_2 \circ (u\star)_1$$

is called the *base change map of rigid cohomology* (with respect to the morphism of frames (2.1.4) and the module E).

Note that our definition of the base change map is slightly different from the definition that can be found in [LS07, Proposition 6.2.6]. Indeed, that definition first applies (2.1.5) and then $\mathrm{Id} \to j^\dagger_{X'}$. It is easy to verify that these definitions amount to the same thing: just write out the compatibility condition for the natural transformation $\mathrm{Id} \to j^\dagger_{X'}$.

The compatibility of rigid cohomology w.r.t. morphisms can easily be understood in terms of base change maps. Indeed, consider a morphism of k-schemes as in the commutative diagram below:

$$\begin{tikzcd} X' \arrow[rr, "f"] \arrow[dr, "p'"'] & & X \arrow[dl, "p"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

Also assume that the morphism of frames (2.1.4) is a realization of f. Now consider an overconvergent F-isocrystal $\mathcal{F}$ on X and let E denote the realization of $\mathcal{F}$ on $(X \subset Y \subset P)$. Then $u^\dagger E$ is the realization of $f^*\mathcal{F}$ on $(X' \subset Y' \subset P')$. The *canonical map*

$$H^i_{rig,C}(X, \mathcal{F}) \longrightarrow H^i_{rig,C'}\left(X', f^*\mathcal{F}\right) \tag{2.1.7}$$

on rigid cohomology is obtained by applying the derived pushforward functor $\mathbb{R}p_{K*}$ to the base change map (2.1.6) and then taking the i-th cohomology. For $\mathcal{F} = \mathcal{O}_{X/K}$ this map expresses the following fact: rigid cohomology with constant coefficients is a functor.

It is also a classical fact that the canonical map (2.1.7) is compatible with the Frobenius action on rigid cohomology. To see this, let $F_{X'}$ and F_X denote the absolute Frobenius on X' resp. on X. Also let $\Phi\colon F_X^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ denote the Frobenius structure on $\mathcal{F}$, where the Frobenius pullback functor is defined as in [LS07, Definition 8.3.1]. Now consider the following diagram:

$$\begin{tikzcd} \mathbb{R}p_{rig,C}\mathcal{F} \arrow[r] \arrow[d] & \mathbb{R}p_{rig,C}F_X^*\mathcal{F} \arrow[r, "\Phi"] \arrow[d] & \mathbb{R}p_{rig,C}\mathcal{F} \arrow[d] \\ \mathbb{R}p'_{rig,C'}f^*\mathcal{F} \arrow[r] & \mathbb{R}p'_{rig,C'}F_{X'}^*f^*\mathcal{F} \arrow[r, "f^*\Phi"] & \mathbb{R}p'_{rig,C'}f^*\mathcal{F} \end{tikzcd}$$

The rows of this diagram describe the Frobenius actions on $\mathbb{R}p_{rig,C}\mathcal{F}$ and on $\mathbb{R}p'_{rig,C'}f^*\mathcal{F}$. The vertical arrows all come from the base change map of the pullback along f. We have to check that this diagram is commutative. All the arrows of the leftmost square are essentially instances of the base change map (2.1.6). This square commutes because of the identity $f \circ F_{X'} = F_X \circ f$. The square on the right commutes because the base change map is compatible with morphisms of overconvergent isocrystals.

2.1.4 Statement of the main theorem

With the material of paragraphs 2.1.1, 2.1.2 and 2.1.3 we are now ready to formulate our main theorem, which is about rigid cohomology with general coefficients.

Theorem 2.1.8. *Let (X', x') and (X, x) be two k-schemes with marked closed points such that $(X', x') \succ (X, x)$ via $f\colon U_{x'} \to X\colon x' \mapsto x$. Let $\mathcal{F} \in$ F-Isoc$^\dagger(X/S)$ be a finitely presented overconvergent F-isocrystal on X. Then for every $i \geq 0$ the canonical map on rigid cohomology*

$$H^i_{rig,\{x\}}(X, \mathcal{F}) \longrightarrow H^i_{rig,\{x'\}}(X', f^*\mathcal{F})$$

is an isomorphism.

Remark 2.1.9.

i) In the statement of the theorem we have implicitly chosen an extension of $f^*\mathcal{F}$ to all of X'. The choice of the extension is not important, since by [LS07, Proposition 8.2.8] the local rigid cohomology only depends on an open neighbourhood of the support.

ii) In the statement of theorem 2.1.8 it is important that x' and x are closed points, otherwise the cohomology with support does not make sense. This is not a problem if one only considers *isolated singularities*, i.e. if one assumes that the singular loci X_{sing} and X'_{sing} are zero-dimensional at x resp. at x'. Indeed, the singular locus of a scheme is closed under specialization; see for example [Liu02, Lemma 2.4.11.(b)]. So for quasi-compact schemes, every isolated singularity is a closed point.

iii) Recall that in the definition of the rigid cohomology with constant coefficients of a scheme X one starts by choosing a realization $(X \subset Y \subset P)$. In order to show that $H^i_{rig}(X)$ is independent of the choice of the realization, one can prove that every diagram

$$\begin{tikzcd} X' = X \arrow[r] \arrow[d, "f = \mathrm{Id}_X"] & Y' \arrow[r] \arrow[d, "g"] & P' \arrow[d, "u"] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

with g proper and u smooth in a neighbourhood of X induces an isomorphism on the cohomology. This is done in [LS07, Proposition 6.5.3]. More specifically, it is the base change map (2.1.6) that induces the isomorphism. Our approach for theorem 2.1.8 is to prove a local version of this result in the case where f is an étale morphism rather than the identity map on X. See proposition 2.2.2 for more details.

We now show that theorem 2.1.1 is just a special case of our main theorem.

Proof of Theorem 2.1.1. Let $\mathcal{O}_{X'/K}$ resp. $\mathcal{O}_{X/K}$ denote the constant F-isocrystal on X' resp. on X. Now choose two morphisms $f'\colon X'' \to X'$ and $f\colon X'' \to X$

that satisfy the conditions from proposition 2.1.2. Then use proposition 2.1.4 together with theorem 2.1.8 and the fact that $(f')^*\mathcal{O}_{X'/K} = f^*\mathcal{O}_{X/K} = \mathcal{O}_{X''/K}$ to obtain an isomorphism

$$H^i_{rig,\{x\}}(X) \xrightarrow{\sim} H^i_{rig,\{x'\}}\left(X'\right). \tag{2.1.8}$$

We have explained in paragraph 2.1.3 that the canonical map on rigid cohomology is Frobenius-equivariant. Therefore the isomorphism (2.1.8) is Frobenius-equivariant as well. □

We end this paragraph with a few more remarks.

Remark 2.1.10. Theorem 2.1.8 can be seen as an analogue of a well-known result of étale cohomology. Indeed, the corresponding statement for the cohomology of an étale sheaf is a special case of the excision theorem [Mil80, Proposition III.1.27]. Combining this with proposition 2.1.2, we see that theorem 2.1.1 also has an analogous statement for ℓ-adic cohomology.

Remark 2.1.11. Frédéric Déglise has pointed out that theorem 2.1.1 can be proved in a more general setting, using the theory of motives. See his remarks in [Dég14].

2.2 Proof of the main theorem

This section contains the proof of our main theorem 2.1.8. First we recall the definition of the canonical map on sheaves with supports. We show that this map is an isomorphism under certain conditions. After this we reformulate our main theorem 2.1.8 in terms of base change maps. We then finish the proof in the last two paragraphs.

2.2.1 The canonical map on sheaves with supports

In this paragraph we give more details about the map $(u\star)_4$ from definition 2.1.7. We show that this map is an isomorphism under a fairly general assumption.

Consider a morphism of frames

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

Recall that if E is an $\mathcal{O}_{]Y[_P}$-module then there is a canonical map

$$u_K^* j_X^\dagger E \longrightarrow j_{X'}^\dagger u_K^* E$$

where $u_K\colon]Y'[_{P'} \to]Y[_P$ denotes the morphism on tubes that one gets by restriction from the morphism $P'_K \to P_K$. This map is an isomorphism if the morphism of frames is Cartesian. See [LS07, Corollary 5.3.9] for more details. We briefly recall how this map can be used to define a canonical map on sheaves with supports. This construction can also be found in [LS07, Corollary 5.3.10]. Let $C' \subset X'$ and $C \subset X$ be closed subschemes such that $f^{-1}(C) \subset C'$. Define $U' = X' \setminus C'$ and $U = X \setminus C$. Then we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & u_K^*\underline{\Gamma}_C^\dagger j_X^\dagger E \arrow[r] & u_K^* j_X^\dagger E \arrow[r] \arrow[d] & u_K^* j_U^\dagger E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\Gamma}_{C'}^\dagger j_{X'}^\dagger u_K^* E \arrow[r] & j_{X'}^\dagger u_K^* E \arrow[r] & j_{U'}^\dagger u_K^* E \arrow[r] & 0
\end{tikzcd} \tag{2.2.1}$$

By the universal property of the kernel we now obtain a canonical morphism

$$u_K^*\underline{\Gamma}_C^\dagger j_X^\dagger E \longrightarrow \underline{\Gamma}_{C'}^\dagger j_{X'}^\dagger u_K^* E. \tag{2.2.2}$$

In the case where E is a $j_X^\dagger \mathcal{O}_{]Y[_P}$-module, composing with $j_{X'}^\dagger$ also gives a canonical map

$$u^\dagger \underline{\Gamma}_C^\dagger E \longrightarrow \underline{\Gamma}_{C'}^\dagger u^\dagger E. \tag{2.2.3}$$

The first step towards proving the main theorem 2.1.8 is to show that the canonical map on sheaves with supports is an isomorphism if the morphism of frames is flat and if the supports are Cartesian.

Proposition 2.2.1. *Let*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\
X \arrow[r] & Y \arrow[r] & P
\end{tikzcd}$$

be a flat morphism of frames. Let E be a $j_X^\dagger \mathcal{O}_{]Y[_P}$-module. Also choose two closed subschemes $C' \subset X'$ and $C \subset X$ such that $C' = C \times_X X'$. Then the canonical map (2.2.3) is an isomorphism.

Proof. We know by [LS07, Corollary 3.3.6] that u_K is flat on some strict neighbourhood V' of $]X'[_{P'}$ in $]Y'[_{P'}$. Therefore, if $j_{V'}$ denotes the inclusion, then the functor $(u_K \circ j_{V'})^*$ is exact. Applying this to the short exact sequence from [LS07, Proposition 5.2.11] yields

$$0 \longrightarrow j_{V'}^{-1} u_K^* \underline{\Gamma}_C^\dagger E \longrightarrow j_{V'}^{-1} u_K^* E \longrightarrow j_{V'}^{-1} u_K^* j_U^\dagger E \longrightarrow 0$$

By using the exactness of the functor $j^\dagger_{X'}$ together with [LS07, Proposition 5.1.13] we obtain the following short exact sequence, which is the same as applying $j^\dagger_{X'}$ to the top row of the diagram (2.2.1):

$$0 \longrightarrow j^\dagger_{X'} u_K^* \underline{\Gamma}^\dagger_C E \longrightarrow j^\dagger_{X'} u_K^* E \longrightarrow j^\dagger_{X'} u_K^* j^\dagger_U E \longrightarrow 0$$

Therefore it is now sufficient to show that the map

$$j^\dagger_{X'} u_K^* j^\dagger_U E \longrightarrow j^\dagger_{U'} u_K^* E \tag{2.2.4}$$

is an isomorphism. Here we have also used that the functor $j^\dagger_{X'}$ is exact, hence preserving kernels. The map (2.2.4) is obtained by applying $j^\dagger_{X'}$ to the map $u_K^* j^\dagger_U E \to j^\dagger_{U'} u_K^* E$ coming from the morphism of frames

$$\begin{tikzcd}
U' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\
U \arrow[r] & Y \arrow[r] & P
\end{tikzcd}$$

This morphism may be factored as

$$\begin{tikzcd}
U' \arrow[d] \arrow[dr] & & \\
U'' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\
U \arrow[r] & Y \arrow[r] & P
\end{tikzcd}$$

where $U'' = U \times_Y Y'$ and where $U' \to U''$ is an open immersion. The map (2.2.4) is then the same thing as applying $j^\dagger_{X'}$ to the composition

$$u_K^* j^\dagger_U E \longrightarrow j^\dagger_{U''} u_K^* E \longrightarrow j^\dagger_{U'} u_K^* E.$$

The first one of these arrows is an isomorphism by [LS07, Corollary 5.3.9]. Applying $j^\dagger_{X'}$ to the second arrow corresponds to the canonical map

$$j^\dagger_{(X' \cap U'')} u_K^* E \longrightarrow j^\dagger_{U'} u_K^* E \tag{2.2.5}$$

coming from the open immersion $U' \hookrightarrow X' \cap U''$. Note that both X' and U'' are open subsets of $X'' = X \times_Y Y'$, which is itself an open subset of Y'. We may now write $U'' = X'' \setminus C''$, where $C'' = C \times_X X''$. But by assumption we have $C' = C \times_X X'$ and therefore $C' = C'' \times_{X''} X'$. This implies that

$$U'' \cap X' = U'' \times_{X''} X' = U'.$$

It follows that the map (2.2.5) is an isomorphism, which finishes the proof. $\square$

2.2.2 Part I of the proof: Reformulation

As we mentioned before, the key to proving our main theorem 2.1.8 is to modify [LS07, Proposition 6.5.3]. The big difference is that in our setting we can only obtain a local result. More specifically, let $x' \in X'$ and $x \in X$ be closed points such that $(X', x') \succ (X, x)$ via some map $f\colon U_{x'} \to X$. Since the local cohomology at x' only depends on an open neighbourhood of x', we may assume that $U_{x'} = X'$ and that f is étale. What we need to show is that for such an f, the base change map with $C' = \{x'\}$ and $C = \{x\}$ is an isomorphism. See proposition 2.2.2 below for the precise statement. If we assume that the morphism of frames in the statement of this proposition is a realization of f and that E is a realization of $\mathcal{F}$ then theorem 2.1.8 follows immediately (c.f. the remarks at the end of paragraph 2.1.3).

Proposition 2.2.2. *Let*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d, "g"] & P' \arrow[d, "u"] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

be a proper smooth morphism of smooth S-frames. Also assume that f is étale. Let E be a coherent $j_X^\dagger \mathcal{O}_{]Y[_P}$-module with an integrable connection over K. Choose two closed points $x' \in X'$ and $x \in X$ such that $f^{-1}(x) = \{x'\}$ and such that f induces an isomorphism $k(x) \xrightarrow{\sim} k(x')$ on the residue fields. Then the base change map

$$u\star\colon \underline{\Gamma}^\dagger_{\{x\}} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K} \longrightarrow (\mathbb{R}u_{K*})\, \underline{\Gamma}^\dagger_{\{x'\}} u^\dagger E \otimes_{\mathcal{O}_{]Y'[_{P'}}} \Omega^\bullet_{]Y'[_{P'}/K}$$

is an isomorphism.

The proof of proposition 2.2.2 will be covered in the next two paragraphs.

2.2.3 Part II of the proof: The quasi-compact étale case

The aim of this paragraph is to prove proposition 2.2.2 in the case of an étale morphism of frames such that the induced morphism on tubes $u_K\colon]Y'[_{P'} \to]Y[_P$ is quasi-compact.

In the case of constant coefficients (i.e. $E = j_X^\dagger \mathcal{O}_{]Y[_P}$) this fact is proved in the appendix of [Ber97b]. Our proof for the general case uses similar techniques as in [Ber97b, Proposition A.10], but introduces two technical improvements. Firstly there is the matter of switching the order of certain functors, which seems to be implicit in [Ber97b]. We will briefly discuss the required properties below. Secondly, some extra care is needed to show that the constructed isomorphism is really the same as the canonical base change map (2.1.6). Indeed, the result in [Ber97b, Proposition A.10] is used to make statements about the *dimension* of rigid cohomology. But for our applications the Frobenius-equivariance is very important.

Proposition 2.2.3. *Let*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

be an étale morphism of smooth S-frames such that the induced morphism on tubes $u_K\colon]Y'[_{P'} \to]Y[_P$ is quasi-compact. Choose two closed points $x' \in X'$ and $x \in X$ such that $f^{-1}(x) = \{x'\}$ and such that f induces an isomorphism $k(x) \xrightarrow{\sim} k(x')$ on the residue fields. Let E be a coherent $j_X^\dagger \mathcal{O}_{]Y[_P}$-module with an integrable connection over K. Then the base change map

$$u\star\colon \underline{\Gamma}^\dagger_{\{x\}} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K} \longrightarrow (\mathbb{R}u_{K*})\, \underline{\Gamma}^\dagger_{\{x'\}} u^\dagger E \otimes_{\mathcal{O}_{]Y'[_{P'}}} \Omega^\bullet_{]Y'[_{P'}/K}$$

is an isomorphism.

Before we can give the proof of this result, we will need to recall the definition of a certain modification of the functor $\underline{\Gamma}^\dagger_C$. This definition also appears in [Ber97b].

Definition 2.2.4. Consider a frame $(X \subset Y \subset P)$ and let $C \subset Y$ be a closed subset. Define $U = X \setminus C$ and $Z = Y \setminus U$. Then for any $\eta < 1$ we may consider the open immersion

$$i_\eta\colon]Y[_P \setminus]Z[_{P\eta} \longrightarrow]Y[_P.$$

As in [Ber97b, A.9], we define

$$\underline{\Gamma}_\eta E = \operatorname{Ker}(E \longrightarrow i_{\eta *} i_\eta^{-1} E)$$

for any $\mathcal{O}_{]Y[_P}$-module E.

We briefly recall some fundamental properties of the functor $\underline{\Gamma}_\eta$. Our formulations are slightly different from the results mentioned in [Ber97b, A.9], because we specialize everything to overconvergent modules.

Proposition 2.2.5. *Use notations as in definition 2.2.4. Then there are canonical isomorphisms*

$$\varinjlim_\eta i_{\eta *} i_\eta^{-1} E \xrightarrow{\sim} j_U^\dagger E. \tag{2.2.6}$$

and

$$\varinjlim_{\eta} \underline{\Gamma}_{\eta}\, j_X^{\dagger} E \xrightarrow{\sim} \underline{\Gamma}_C^{\dagger} j_X^{\dagger} E. \tag{2.2.7}$$

Proof. The isomorphism (2.2.6) is proved in [Ber97b, A.9]. The isomorphism (2.2.7) follows by combining (2.2.6) with the short exact sequence from [LS07, Proposition 5.2.11]. Here one also uses that filtered colimits commute with finite limits. □

For the next property we focus on the case where the support C is a closed point.

Proposition 2.2.6. *Use notations as in definition 2.2.4. Assume that $C = \{x\}$ is a closed point. Write $W =]\{x\}[_P$ and let $i_W \colon W \hookrightarrow]Y[_P$ denote the inclusion map. Then for any $\eta < 1$ and for any $\mathcal{O}_{]Y[_P}$-module E, the base change map*

$$\underline{\Gamma}_{\eta}\, j_X^{\dagger} E \longrightarrow (\mathbb{R} i_{W*})\, i_W^{*} \underline{\Gamma}_{\eta}\, j_X^{\dagger} E \tag{2.2.8}$$

that is associated to the diagram

$$\begin{tikzcd} W \arrow[r, "i_W"] \arrow[d, "i_W"] &]Y[_P \arrow[d, "Id"] \\ ]Y[_P \arrow[r, "Id"] &]Y[_P \end{tikzcd}$$

is an isomorphism.

Proof. Consider the immersion $\iota \colon]Z[_P \to]Y[_P$. It is proved in [Ber97b, A.9] that the canonical map

$$\underline{\Gamma}_{\eta}\, j_X^{\dagger} E \longrightarrow (\mathbb{R}\iota_*)\, \iota^* \underline{\Gamma}_{\eta}\, j_X^{\dagger} E$$

is an isomorphism. Therefore it suffices to show that there is an identification

$$(\mathbb{R}\iota_*)\, \iota^* \underline{\Gamma}_{\eta}\, j_X^{\dagger} E \cong (\mathbb{R} i_{W*})\, i_W^{*} \underline{\Gamma}_{\eta}\, j_X^{\dagger} E. \tag{2.2.9}$$

To see this, we use that $\{x\} \subset Y$ is a closed subset. This allows us to write, according to [LS07, Proposition 2.2.15]:

$$]Z[_P =]Y \setminus X[_P \cup]\{x\}[_P =]Y \setminus X[_P \cup W.$$

But by construction, the restriction of $j_X^{\dagger} E$ to $]Y \setminus X[_P$ is zero. The identification (2.2.9) follows. □

Next we investigate how the functor $\underline{\Gamma}_{\eta}$ behaves w.r.t. morphisms of frames.

Definition 2.2.7. Consider a morphism of frames

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d, "g"] & P' \arrow[d, "u"] \\
X \arrow[r] & Y \arrow[r] & P
\end{tikzcd}$$

together with supports $C' \subset X'$ and $C \subset X$ such that $f^{-1}(C) \subset C'$. Define $U' = X' \setminus C'$ and $U = X \setminus C$, and note that f restricts to $f\colon U' \to U$. Also consider closed complements $Z' = Y' \setminus U'$ and $Z = Y \setminus U$, which satisfy $g^{-1}(Z) \subset Z'$.

Then we have the following commutative diagram, for any $\eta < 1$:

$$\begin{tikzcd}
]Y'[_{P'} \backslash]Z'[_{P'\eta} \arrow[r, "i'_\eta"] \arrow[d] &]Y'[_{P'} \arrow[d, "u_K"] \\
]Y[_P \backslash]Z[_{P\eta} \arrow[r, "i_\eta"] &]Y[_P
\end{tikzcd}$$

For an $\mathcal{O}_{]Y[_P}$-module E we may then consider the (underived) base change map:

$$u_K^* i_{\eta*} i_\eta^{-1} E \longrightarrow i'_{\eta*} u_K^* i_\eta^{-1} E = i'_{\eta*} (i'_\eta)^{-1} u_K^* E.$$

Then, using the left exactness of $j^\dagger_{X'}$, we obtain a commutative diagram whose top row is a complex and whose bottom row is exact:

$$\begin{tikzcd}
 & j^\dagger_{X'} u_K^* \underline{\Gamma}_\eta E \arrow[r] & j^\dagger_{X'} u_K^* E \arrow[r] \arrow[d] & j^\dagger_{X'} u_K^* i_{\eta*} i_\eta^{-1} E \arrow[d] \\
0 \arrow[r] & j^\dagger_{X'} \underline{\Gamma}'_\eta u_K^* E \arrow[r] & j^\dagger_{X'} u_K^* E \arrow[r] & j^\dagger_{X'} i'_{\eta*} (i'_\eta)^{-1} u_K^* E
\end{tikzcd} \tag{2.2.10}$$

By the universal property of the kernel we now obtain a map

$$j^\dagger_{X'} u_K^* \underline{\Gamma}_\eta E \longrightarrow j^\dagger_{X'} \underline{\Gamma}'_\eta u_K^* E. \tag{2.2.11}$$

Finally, by passing to the limit $\eta \to 1$, one obtains a map

$$\varinjlim_\eta j^\dagger_{X'} u_K^* \underline{\Gamma}_\eta E \longrightarrow \varinjlim_\eta j^\dagger_{X'} \underline{\Gamma}'_\eta u_K^* E. \tag{2.2.12}$$

Proposition 2.2.8. *Use the notations from definition 2.2.7 and assume that the morphism of frames is flat. Then the top row of the diagram (2.2.10) is exact. Also, for any $\eta < 1$, the object $j^\dagger_{X'} u_K^* \underline{\Gamma}_\eta E$ is isomorphic to the kernel of the map*

$$j^\dagger_{X'} u_K^* E \longrightarrow j^\dagger_{X'} u_K^* i_{\eta*} i_\eta^{-1} E.$$

Proof. We know by [LS07, Corollary 3.3.6] that u_K is flat on some strict neighbourhood V' of $]X'[_{P'}$ in $]Y'[_{P'}$. Therefore, if $j_{V'}$ denotes the inclusion, then the functor $(u_K \circ j_{V'})^*$ is exact. This results in a short exact sequence

$$0 \longrightarrow j_{V'}^{-1} u_K^* \underline{\Gamma}_\eta E \longrightarrow j_{V'}^{-1} u_K^* E \longrightarrow j_{V'}^{-1} u_K^* i_{\eta *} i_\eta^{-1} E$$

By applying the left exact functor $j_{X'}^\dagger j_{V'*}$ and using [LS07, Proposition 5.1.13], we indeed find an exact sequence

$$0 \longrightarrow j_{X'}^\dagger u_K^* \underline{\Gamma}_\eta E \longrightarrow j_{X'}^\dagger u_K^* E \longrightarrow j_{X'}^\dagger u_K^* i_{\eta *} i_\eta^{-1} E$$

This finishes the proof. □

The previous proposition is useful for the following lemma, which is one of the main technical ingredients in the proof of proposition 2.2.11 below.

Proposition 2.2.9. *Use the notations from definition 2.2.7. Assume that the morphism of frames is flat and that E is a $j_X^\dagger \mathcal{O}_{]Y[_P}$-module. Then the map (2.2.12) is isomorphic to the map (2.2.3).*

Proof. By considering the constructions in the proofs of propositions 2.2.8 and 2.2.1, it is sufficient to show the following statement: The map

$$\varinjlim_\eta j_{X'}^\dagger u_K^* i_{\eta *} i_\eta^{-1} E \longrightarrow \varinjlim_\eta j_{X'}^\dagger i'_{\eta *} (i'_\eta)^{-1} u_K^* E$$

coming from the rightmost vertical arrow of diagram (2.2.10) is the same thing as the map

$$j_{X'}^\dagger u_K^* j_U^\dagger E \longrightarrow j_{U'}^\dagger u_K^* E \tag{2.2.13}$$

coming from the rightmost vertical arrow of diagram (2.2.1).

On modules, the functor $j_{X'}^\dagger$ is left adjoint to the forgetful functor, according to [LS07, Proposition 5.3.1]. This means that $j_{X'}^\dagger$ preserves colimits, so that we are reduced to the map

$$j_{X'}^\dagger \varinjlim_\eta u_K^* i_{\eta *} i_\eta^{-1} E \longrightarrow j_{X'}^\dagger \varinjlim_\eta i'_{\eta *} (i'_\eta)^{-1} u_K^* E.$$

The limit $\eta \to 1$ also slides through u_K^*, which is a left adjoint. By proposition 2.2.5 we indeed recover the map (2.2.13). □

In the remainder of this paragraph we will consider the map

$$j_{X'}^\dagger u_K^* \underline{\Gamma}_\eta E \longrightarrow j_{X'}^\dagger \underline{\Gamma}'_\eta u_K^* E \longrightarrow \underline{\Gamma}'_\eta j_{X'}^\dagger u_K^* E \tag{2.2.14}$$

where the first arrow is the map (2.2.11) and the second arrow is obtained by applying $j_{X'}^\dagger \underline{\Gamma}'_\eta$ to the map $u_K^* E \to j_{X'}^\dagger u_K^* E$. Here we also use that the module $\underline{\Gamma}'_\eta j_{X'}^\dagger u_K^* E$ is already overconvergent, so there is no need to write the $j_{X'}^\dagger$ in front of it. Using our earlier results we easily obtain the following property.

Proposition 2.2.10. *Again use the notations from definition 2.2.7. Assume that the following conditions hold:*

i) The given morphism of frames is flat.

ii) E is a $j_X^\dagger \mathcal{O}_{]Y[_P}$-module.

iii) The supports satisfy $f^{-1}(C) = C'$.

Then the map (2.2.14) becomes an isomorphism when passing to the limit $\eta \to 1$.

Proof. For the map (2.2.12) this follows immediately by combining propositions 2.2.9 and 2.2.1. Using the previous techniques, it is easy to show that

$$j_{X'}^\dagger \underline{\Gamma}'_\eta u_K^* E \longrightarrow \underline{\Gamma}'_\eta j_{X'}^\dagger u_K^* E$$

becomes the identity on $\underline{\Gamma}_{C'}^\dagger j_{X'}^\dagger u_K^* E$ in the limit $\eta \to 1$. ☐

We now use the results about the functor $\underline{\Gamma}_\eta$ and the map (2.2.14) to prove a weak version of proposition 2.2.3.

Proposition 2.2.11. *Consider an étale morphism of frames*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

such that the induced morphism on tubes $u_K\colon]Y'[_{P'} \to]Y[_P$ is quasi-compact. Choose two closed points $x' \in X'$ and $x \in X$ such that $f^{-1}(x) = \{x'\}$ and such that f induces an isomorphism $k(x) \xrightarrow{\sim} k(x')$ on the residue fields. Let E be a coherent $j_X^\dagger \mathcal{O}_{]Y[_P}$-module. Then the canonical map

$$\underline{\Gamma}_{\{x\}}^\dagger E \longrightarrow (\mathbb{R}u_{K*})\, u_K^* \underline{\Gamma}_{\{x\}}^\dagger E \longrightarrow (\mathbb{R}u_{K*})\, u^\dagger \underline{\Gamma}_{\{x\}}^\dagger E \tag{2.2.15}$$

that is defined in a similar way as the composition $(u\star)_2 \circ (u\star)_1$ from paragraph 2.1.3 is an isomorphism.

Proof. Define $W' =]\{x'\}[_{P'}$ and $W =]\{x\}[_P$. Then we have a commutative diagram

$$\begin{tikzcd} W' \arrow[r, "i_{W'}"] \arrow[d, "v"] &]Y'[_{P'} \arrow[d, "u_K"] \\ W \arrow[r, "i_W"] &]Y[_P \end{tikzcd}$$

where v is the restriction of u_K and $i_{W'}$, i_W are open immersions. The fact that f induces an isomorphism $k(x) \xrightarrow{\sim} k(x')$ on residue fields means that the restriction $f\colon \operatorname{Spec} k(x') \to \operatorname{Spec} k(x)$ is an isomorphism. Since u_K is étale in a neighbourhood of x' it then follows from [LS07, Proposition 2.3.15] that v is an isomorphism.

We now apply a standard property about the behaviour of the base change map (2.1.3) with respect to a composition of diagrams. This property is formulated for schemes in [SGA4, Proposition XII.4.4]. The proof is formally the same for any ringed space. We will apply this composition property to the diagram

$$\begin{tikzcd} W' \arrow[r, "i_{W'}"] \arrow[d, "v"] &]Y'[_{P'} \arrow[r, "u_K"] \arrow[d, "u_K"] &]Y[_P \arrow[d, "\mathrm{Id}"] \\ W \arrow[r, "i_W"] &]Y[_P \arrow[r, "\mathrm{Id}"] &]Y[_P \end{tikzcd} \tag{2.2.16}$$

and to the sheaf $i_W^* \underline{\Gamma}_\eta E$. By making use of the isomorphism from proposition 2.2.6 we obtain a commutative diagram

$$\begin{tikzcd} \underline{\Gamma}_\eta E \arrow[r, "a_1"] \arrow[dr, "a_3"'] & (\mathbb{R}u_{K*})\, u_K^* \underline{\Gamma}_\eta E \arrow[d, "a_2"] \\ & (\mathbb{R}u_{K*} i_{W'*})\, v^* i_W^* \underline{\Gamma}_\eta E \end{tikzcd}$$

The horizontal arrow a_1 is the base change map coming from the rightmost square of (2.2.16), applied to the sheaf $(\mathbb{R}i_{W*})\, i_W^* \underline{\Gamma}_\eta E$. The map a_2 is obtained by applying $\mathbb{R}u_{K*}$ to the base change map coming from the leftmost square of (2.2.16). By using [SGA4, Proposition XII.4.4] again we see that a_2 is equal to the morphism that one gets after applying $\mathbb{R}u_{K*}$ to the canonical morphism

$$u_K^* \underline{\Gamma}_\eta E \longrightarrow (\mathbb{R}i_{W'*})\, i_{W'}^* u_K^* \underline{\Gamma}_\eta E.$$

The arrow a_3 is obtained by taking the base change map of the total diagram (2.2.16). Note that a_3 is equal to the morphism that one gets by applying $\mathbb{R}i_{W*}$ to the canonical map

$$i_W^* \underline{\Gamma}_\eta E \longrightarrow (\mathbb{R}v_*)\, v^* i_W^* \underline{\Gamma}_\eta E.$$

Since v is an isomorphism it follows that a_3 is an isomorphism. Let us now look at the diagram

$$
\begin{array}{ccc}
(\mathbb{R}u_{K*})\, u_K^* \underline{\Gamma}_\eta E & \xrightarrow{\quad b_1 \quad} & (\mathbb{R}u_{K*})\, j_{X'}^\dagger u_K^* \underline{\Gamma}_\eta E \\
\Big\downarrow {\scriptstyle a_2} & & \Big\downarrow {\scriptstyle b_2} \\
(\mathbb{R}u_{K*} i_{W'*})\, i_{W'}^* u_K^* \underline{\Gamma}_\eta E & \xrightarrow[b_3]{} & (\mathbb{R}u_{K*} i_{W'*})\, i_{W'}^* j_{X'}^\dagger u_K^* \underline{\Gamma}_\eta E
\end{array}
$$

that is obtained by applying the canonical map

$$
\mathrm{Id} \longrightarrow (\mathbb{R}i_{W'*})\, i_{W'}^*
$$

to the morphism

$$
u_K^* \underline{\Gamma}_\eta E \longrightarrow j_{X'}^\dagger u_K^* \underline{\Gamma}_\eta E
$$

and then composing with $\mathbb{R}u_{K*}$.

Note that the map b_3 is an isomorphism. This follows from the characterization [LS07, Proposition 5.1.12] of the functor $j_{X'}^\dagger$. Indeed, $i_{W'}^*$ is a left adjoint, hence preserving filtered colimits. If V' is any strict neighbourhood of $]X'[_{P'}$ in $]Y'[_{P'}$, then $W' \subset V'$, so that $i_{W'}^* j_{V'*} j_{V'}^{-1} = i_{W'}^*$. This shows that $i_{W'}^* j_{X'}^\dagger = i_{W'}^*$.

In a similar way we construct another diagram

$$
\begin{array}{ccc}
(\mathbb{R}u_{K*})\, j_{X'}^\dagger u_K^* \underline{\Gamma}_\eta E & \xrightarrow{\quad c_1 \quad} & (\mathbb{R}u_{K*})\, \underline{\Gamma}'_\eta\, j_{X'}^\dagger u_K^* E \\
\Big\downarrow {\scriptstyle b_2} & & \Big\downarrow {\scriptstyle c_2} \\
(\mathbb{R}u_{K*} i_{W'*})\, i_{W'}^* j_{X'}^\dagger u_K^* \underline{\Gamma}_\eta E & \xrightarrow[c_3]{} & (\mathbb{R}u_{K*} i_{W'*})\, i_{W'}^* \underline{\Gamma}'_\eta\, j_{X'}^\dagger u_K^* E
\end{array}
$$

using the morphism (2.2.14). Observe that the map c_2 is an isomorphism, according to proposition 2.2.6.

We now have the identity

$$
c_2 \circ c_1 \circ b_1 \circ a_1 = c_3 \circ b_3 \circ a_3 \tag{2.2.17}
$$

and we have shown that a_3, b_3 and c_2 are isomorphisms.

At this point, we fix an $m \geq 0$ and we consider the m-th cohomology of all the complexes above. It is sufficient to consider these modules, since the statement of the proposition is about a complex concentrated in degree zero. By abuse of notation, we still write the maps as $a_1, b_1, \ldots$.

Note that (the m-th cohomology of) the map (2.2.15) is recovered by taking the limit $\eta \to 1$ of the composition $b_1 \circ a_1$. To see this, first observe that $\varinjlim_\eta$ commutes with $\mathbb{R}^m u_{K*}$, since u_K is assumed quasi-compact. See [Ber96, 0.1.8] for details. Also, $\varinjlim_\eta$ commutes with the functors u_K^* and $j_{X'}^\dagger$, since these are left adjoints.

We have proved in proposition 2.2.10 that c_1 becomes an isomorphism in the limit $\eta \to 1$. We now show that c_3 has the same property. By the

Grothendieck spectral sequence is suffices to show that for every $n \geq 0$, the map

$$(\mathbb{R}^m u_{K*})\,(\mathbb{R}^n i_{W'*})\, i^*_{W'} j^\dagger_{X'} u^*_K \underline{\Gamma}_\eta E \longrightarrow (\mathbb{R}^m u_{K*})\,(\mathbb{R}^n i_{W'*})\, i^*_{W'} \underline{\Gamma}'_\eta\, j^\dagger_{X'} u^*_K E \tag{2.2.18}$$

becomes an isomorphism for $\eta \to 1$. To see this, we use that W' is a union of nested quasi-compact sets:

$$W' = \bigcup_{\lambda<1} W'_\lambda, \tag{2.2.19}$$

where $W'_\lambda = [\{x'\}]_{P'\lambda}$. Now consider the following map, for $\eta < 1$ and $\lambda < 1$:

$$\begin{aligned} f_{\eta\,\lambda}\colon\ (\mathbb{R}^m u_{K*}) \left(\mathbb{R}^n i_{W'_\lambda *}\right) i^*_{W'_\lambda} j^\dagger_{X'} u^*_K \underline{\Gamma}_\eta E& \\ \longrightarrow (\mathbb{R}^m u_{K*}) \left(\mathbb{R}^n i_{W'_\lambda *}\right) i^*_{W'_\lambda} \underline{\Gamma}'_\eta\, j^\dagger_{X'} u^*_K E&. \end{aligned} \tag{2.2.20}$$

Then consider the colimit over the diagram indexed by $]0,1[\times]0,1[$. By general category theory one knows that this colimit may be computed by first considering the colimit over one of the sets, then the colimit over the other set.

Let us first consider the map $\varinjlim_\eta \varinjlim_\lambda f_{\eta\,\lambda}$. For some sheaf $\mathcal{F}$ we have for every $\lambda < 1$ a map

$$(\mathbb{R}^n i_{W'*})\, i^*_{W'} \mathcal{F} \longrightarrow \left(\mathbb{R}^n i_{W'_\lambda *}\right) i^*_{W'_\lambda} \mathcal{F}.$$

We argue that this map becomes an isomorphism in the limit $\lambda \to 1$. This can be verified on stalks: start by choosing a point Q. Then the stalk at Q of $\varinjlim_\lambda \left(\mathbb{R}^n i_{W'_\lambda *}\right) i^*_{W'_\lambda} \mathcal{F}$ is given by:

$$\varinjlim_\lambda \varinjlim_{Q\in U} H^n(U \cap W'_\lambda, \mathcal{F}). \tag{2.2.21}$$

There are two cases to consider. If $Q \in W'$ then we have $Q \in W'_\lambda$ for λ large enough. Then $\{U \cap W'_\lambda\}$ is cofinal in the set of all neighbourhoods $Q \in U$, and we find:

$$\varinjlim_\lambda \varinjlim_{Q\in U} H^n(U \cap W'_\lambda, \mathcal{F}) = \varinjlim_{Q\in U} H^n(U, \mathcal{F}).$$

In the other case we have $Q \notin W'$, and the limit (2.2.21) is zero. In both cases we recover the stalk at Q of the sheaf $(\mathbb{R}^n i_{W'*})\, i^*_{W'} \mathcal{F}$. As a result (also using the fact that u_K is quasi-compact), we find that $\varinjlim_\eta \varinjlim_\lambda f_{\eta\,\lambda}$ is isomorphic to

$$\varinjlim_\eta (\mathbb{R}^m u_{K*})\,(\mathbb{R}^n i_{W'*})\, i^*_{W'} j^\dagger_{X'} u^*_K \underline{\Gamma}_\eta E \longrightarrow \varinjlim_\eta (\mathbb{R}^m u_{K*})\,(\mathbb{R}^n i_{W'*})\, i^*_{W'} \underline{\Gamma}'_\eta\, j^\dagger_{X'} u^*_K E,$$

which is precisely the map that we wish to study. Next we compute the colimit over $]0,1[\times]0,1[$ by considering $\varinjlim_{\lambda} \varinjlim_{\eta} f_{\eta\,\lambda}$. Since W'_λ is quasi-compact for every $\lambda < 1$, this is isomorphic to

$$\varinjlim_{\lambda} (\mathbb{R}^m u_{K*}) \left(\mathbb{R}^n i_{W'_\lambda *}\right) i^*_{W'_\lambda} \varinjlim_{\eta} j^\dagger_{X'} u^*_K \underline{\Gamma}_\eta E \\ \longrightarrow \varinjlim_{\lambda} (\mathbb{R}^m u_{K*}) \left(\mathbb{R}^n i_{W'_\lambda *}\right) i^*_{W'_\lambda} \varinjlim_{\eta} \underline{\Gamma}'_\eta j^\dagger_{X'} u^*_K E.$$

By proposition 2.2.10, we see that this is an isomorphism. As a result, the limit $\eta \to 1$ of the map (2.2.18) is an isomorphism.

Once we know that c_3 is an isomorphism in the limit, it follows immediately that also $b_1 \circ a_1$ becomes an isomorphism for $m = 0$ and $\eta \to 1$.

Similarly, for $m > 0$ we know that

$$(\mathbb{R}^m u_{K*})\, i_{W'*} i^*_{W'} u^*_K \underline{\Gamma}_\eta E = 0.$$

But in the limit $\eta \to 1$ this module is isomorphic to

$$\varinjlim_{\eta} (\mathbb{R}^m u_{K*})\, j^\dagger_{X'} u^*_K \underline{\Gamma}_\eta E = (\mathbb{R}^m u_{K*})\, j^\dagger_{X'} u^*_K \underline{\Gamma}^\dagger_{\{x\}} E,$$

which must also be zero. This completes the proof. □

With proposition 2.2.11 in place the proof of proposition 2.2.3 becomes relatively straightforward.

Proof of Proposition 2.2.3. As we did in paragraph 2.1.3, we divide the base change map $u\star$ into several parts $(u\star)_i$ for $i = 1, \ldots, 4$. It follows from proposition 2.2.1 that $(u\star)_4$ is an isomorphism. By [LS07, Corollary 3.3.6] we know that the map $u_K \colon]Y'[_{P'} \to]Y[_P$ is étale on some strict neighbourhood V' of $]X'[_{P'}$ in $]Y'[_{P'}$. It then follows from [LS07, Proposition 5.3.7] that $(u\star)_3$ is an isomorphism if and only if the corresponding map for the restriction $u_K \colon V' \to]Y[_P$ is an isomorphism. Therefore we may work as if u_K were étale. Since our frames are smooth, we may also work as if $]Y'[_{P'}$ and $]Y[_P$ were smooth. In this situation the canonical map $u^*_K \Omega^1_{]Y[_P} \to \Omega^1_{]Y'[_{P'}}$ is an isomorphism. It immediately follows that $(u\star)_3$ is an isomorphism as well.

It remains to show that the composition $(u\star)_2 \circ (u\star)_1$ is an isomorphism. For this we can use proposition 2.2.11. Indeed, proposition 2.2.11 is equivalent to saying that for every coherent $j^\dagger_X \mathcal{O}_{]Y[_P}$-module $\mathcal{E}$, the map

$$\underline{\Gamma}^\dagger_{\{x\}} \mathcal{E} \longrightarrow u_{K*} u^\dagger \underline{\Gamma}^\dagger_{\{x\}} \mathcal{E} \tag{2.2.22}$$

is an isomorphism and

$$(\mathbb{R}^m u_{K*})\, u^\dagger \underline{\Gamma}^\dagger_{\{x\}} \mathcal{E} = 0 \tag{2.2.23}$$

for all $m > 0$. Note that by Proposition 5.3.2 and Corollary 5.3.3 in [LS07], every term of the de Rham complex $\underline{\Gamma}^\dagger_{\{x\}} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K}$ satisfies the conditions of proposition 2.2.11. By using the spectral sequence of hypercohomology

$$E_1^{m,n} = (\mathbb{R}^m u_{K*})\, u^\dagger \left(\underline{\Gamma}^\dagger_{\{x\}} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^n_{]Y[_P/K}\right) \Longrightarrow \\ \left(\mathbb{R}^{m+n} u_{K*}\right) u^\dagger \left(\underline{\Gamma}^\dagger_{\{x\}} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K}\right)$$

we deduce from (2.2.23) that

$$(\mathbb{R} u_{K*})\, u^\dagger \left(\underline{\Gamma}^\dagger_{\{x\}} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K}\right) = u_{K*} u^\dagger \left(\underline{\Gamma}^\dagger_{\{x\}} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K}\right).$$

This means that $(u\star)_2 \circ (u\star)_1$ is equal to the canonical map

$$\underline{\Gamma}^\dagger_{\{x\}} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K} \longrightarrow u_{K*} u^\dagger \left(\underline{\Gamma}^\dagger_{\{x\}} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K}\right). \tag{2.2.24}$$

But this map can be computed term by term. From the fact that (2.2.22) is an isomorphism it then follows that (2.2.24) is an isomorphism as well. We have now shown that the maps $(u\star)_4$, $(u\star)_3$ and $(u\star)_2 \circ (u\star)_1$ are isomorphisms. This finishes the proof. □

Remark 2.2.12. We have shown that the composition $(u\star)_2 \circ (u\star)_1$ is an isomorphism. However, the individual maps $(u\star)_2$ and $(u\star)_1$ need not be isomorphisms. For this reason it was convenient to make definition 2.1.7 slightly different from the definition in [LS07].

2.2.4 Part III of the proof: The general case

In this paragraph we finish the proof of proposition 2.2.2. First we improve proposition 2.2.3 by removing the condition that the induced map on tubes $u_K\colon]Y'[_{P'} \to]Y[_P$ should be quasi-compact.

Proposition 2.2.13. *Let*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

be an étale morphism of smooth S-frames. Also assume that f is étale. Let E be a coherent $j_X^\dagger \mathcal{O}_{]Y[_P}$-module with an integrable connection over K. Choose two closed points $x' \in X'$ and $x \in X$ such that $f^{-1}(x) = \{x'\}$ and such that f defines an isomorphism $k(x) \xrightarrow{\sim} k(x')$ on the residue fields. Then the base change map

$$u\star\colon \underline{\Gamma}^\dagger_{\{x\}} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^\bullet_{]Y[_P/K} \longrightarrow (\mathbb{R} u_{K*})\, \underline{\Gamma}^\dagger_{\{x'\}} u^\dagger E \otimes_{\mathcal{O}_{]Y'[_{P'}}} \Omega^\bullet_{]Y'[_{P'}/K}$$

is an isomorphism.

Proof. Let $Q' \subset P'$ be an open neighbourhood of X' such that the restriction $u_{|Q'}$ is étale. Then define $X'' = X \times_P Q'$ and $Y'' = Y \times_P P'$. We claim that the canonical map $X' \to Y''$ is an open immersion. Since this morphism factors through X'' and since it is easy to see that the canonical map $X'' \to Y''$ is an open immersion, it suffices to show that the canonical morphism $\alpha \colon X' \to X''$ is an open immersion. It is clear that α is an immersion. Now consider the projection morphism $\beta \colon X'' \to X$. Since $f = \beta \circ \alpha$ and since β is étale by construction, it follows that α is étale as well. This shows that α is indeed an open immersion, proving our claim. The fact that $X' \to Y''$ is an open immersion allows us to factor our morphism of frames as follows:

$$\begin{tikzcd} & Y' \arrow[d] \arrow[dr] & \\ X' \arrow[ur] \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

Recall that we assumed the formal schemes P' and P to be topologically of finite type over $\mathcal{V}$. It follows that these formal schemes are Noetherian topological spaces, since their closed fibers are of finite type over k. In particular we see that u is quasi-compact. Therefore the morphism $u_K \colon P'_K \to P_K$ on the generic fibers is quasi-compact as well. Since the rightmost square in the diagram above is Cartesian we find that $u_K^{-1}(]Y[_P) =]Y''[_{P'}$ according to [LS07, Proposition 2.2.6]. It follows that the induced morphism on tubes $u_K \colon]Y''[_{P'} \to]Y[_P$ is quasi-compact. This allows us to apply proposition 2.2.3 to the lower part of the diagram. Now observe that the morphism $Y' \to Y''$ is a closed immersion, hence proper. So according to [LS07, Proposition 6.5.3], the base change map that is associated to the upper part of the diagram is an isomorphism as well. $\square$

The key idea for the proof of proposition 2.2.2 is to show that an étale map $f \colon X' \to X$ has an étale realization, at least after shrinking X' and X. In this way one reduces the problem to proposition 2.2.13. The proof of this fact relies on a number of geometric results that we discuss below.

Proposition 2.2.14.

i) Consider a proper morphism of frames

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d] & P' \arrow[d] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

where f is quasi-projective. Then we can blow up a closed subvariety of Y' outside X' in P' to obtain a diagram

$$\begin{tikzcd} & \widetilde{Y'} \arrow[r] \arrow[d] & \widetilde{P'} \arrow[d] \\ X' \arrow[ur] \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d] & P' \arrow[d] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

where the composition $\widetilde{Y'} \to Y$ is projective.

ii) Consider a strict morphism of frames

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & P' \arrow[d, "u"] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

where u is a formal blowing up. Then the map $u_K \colon]Y'[_{P'} \to]Y[_P$ is an isomorphism. Moreover, any admissible open neighbourhood V of $]Y[_P$ is a strict neighbourhood of $]X[_P$ in $]Y[_P$ if and only if $u_K^{-1}(V)$ is a strict neighbourhood of $]X'[_{P'}$ in $]Y'[_{P'}$. That is, giving a $j_X^\dagger \mathcal{O}_{]Y[_P}$-module amounts to the same thing as giving a $j_{X'}^\dagger \mathcal{O}_{]Y'[_{P'}}$-module.

iii) Consider a frame $(X \subset Y \subset P)$ together with a diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & Y' \arrow[d, "g"] & \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

where the map $X' \to Y'$ is an open immersion, f is an étale morphism and g is projective. Then locally on $(X \subset Y \subset P)$, there exists a closed subscheme $Y'' \subset Y'$ containing X' such that the map $g_{|Y''}$ extends to a proper étale morphism of frames

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f"] & Y'' \arrow[r] \arrow[d, "g"] & P' \arrow[d, "u"] \\
X \arrow[r] & Y \arrow[r] & P
\end{tikzcd}$$

Proof.

i) Apply [RG71, Corollaire 5.7.14] to the morphism $Y' \to Y$ and the open subset $X' \subset Y'$, which is quasi-projective over Y.

ii) This follows from [LS07, Corollary 2.2.7] and [LS07, Proposition 3.1.13].

iii) The composition $Y' \to Y \to P$ can be factored through a closed immersion $Y' \to \mathbb{P}^N_P$ for some N. It now suffices to show that the morphism i in the diagram below is a regular immersion. The rest of the proof is analogous to [LS07, Lemma 6.5.1].

$$\begin{tikzcd}
X' \arrow[dr, "i"] \arrow[drr, bend left] \arrow[ddr, bend right, "f"'] & & \\
& \mathbb{P}^N_P \times_P X \arrow[r] \arrow[d] & \mathbb{P}^N_P \arrow[d] \\
& X \arrow[r] & P
\end{tikzcd}$$

First note that i is an immersion, since the morphisms $X' \to \mathbb{P}^N_P$ and $\mathbb{P}^N_P \times_P X \to \mathbb{P}^N_P$ are immersions. Also, $\mathbb{P}^N_P \times_P X \to X$ is smooth since it is obtained by base extension from a smooth morphism. Since f is a local complete intersection morphism it follows from [Liu02, Corollary 6.3.22] that i is indeed regular.

□

With all the preliminary work, the proof of proposition 2.2.2 becomes very similar to the proof of [LS07, Proposition 6.5.3].

Proof of Proposition 2.2.2. First note that we can always replace X' and X by open neighbourhoods $U_{x'}$, U_x of x' resp. of x such that $f(U_{x'}) \subset U_x$. Indeed, the base change map coming from the diagram

$$\begin{tikzcd}
U_x \arrow[d] \arrow[dr] & & \\
X \arrow[r] & Y \arrow[r] & P
\end{tikzcd}$$

is simply the canonical map $\underline{\Gamma}^\dagger_{\{x\}} \to \underline{\Gamma}^\dagger_{\{x\}} j^\dagger_{U_x}$ applied to the de Rham complex of E. This is an isomorphism by [LS07, Proposition 5.2.12]. A similar argument holds for the inclusion $U_{x'} \hookrightarrow X'$ and the $j^\dagger_{X'}\mathcal{O}_{]Y'[_{P'}}$-module $u^\dagger E$. We may also replace Y' by a closed subscheme that contains X'. By [LS07, Proposition 6.5.3] this does not alter the base change map either. We will refer to a combination of these two operations as a *shrinking* of the data. The fact that a shrinking of the data does not alter the base change map can be used to reduce the problem to the case where f has an étale realization. Indeed, after replacing X' and X by open neighbourhoods of x' resp. of x we may assume that f is an affine morphism, hence quasi-projective. By the first two points of proposition 2.2.14 we then reduce to the case where g is projective. After some more shrinking of X' and Y' we may use the third point of proposition 2.2.14 to obtain an étale morphism of frames

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & Y' \arrow[r] \arrow[d, "g"] & P'' \arrow[d, "v"] \\ X \arrow[r] & Y \arrow[r] & P \end{tikzcd}$$

Now consider the diagonal embedding $Y' \hookrightarrow P''' = P' \times_P P''$ and let $p_1 \colon P''' \to P'$ and $p_2 \colon P''' \to P''$ denote the projection maps. By construction we have that

$$u \circ p_1 = v \circ p_2. \tag{2.2.25}$$

Also, p_1 and p_2 are smooth since they are obtained by base extension from v resp. from u. By the identity (2.2.25) it is now sufficient to prove that the base change maps that are associated to v and to the diagrams

$$\begin{tikzcd} & & P''' \arrow[dd, "p_1, p_2"] \\ X' \arrow[r] & Y' \arrow[ur] \arrow[dr] & \\ & & P', P'' \end{tikzcd}$$

are isomorphisms. In proposition 2.2.13 we have already proved that the base change map for the étale morphism of frames v is an isomorphism. For the two morphisms of frames p_1 and p_2 it follows directly from [LS07, Proposition 6.5.3]. $\square$

Chapter 3

The local cohomology of a weighted homogeneous singularity

In this chapter we will study the local cohomology of a weighted homogeneous hypersurface singularity, i.e. a singularity that is given by a weighted homogeneous equation for a suitable choice of local coordinates. The aim is to show that under certain assumptions the local cohomology of such a singularity is related to the rigid cohomology of a certain smooth projective hypersurface.

We give the necessary definitions in paragraphs 3.1.1 and 3.1.2 below. The main result for this chapter is formulated in paragraph 3.1.3. The proof of this result can be naturally broken down into several statements, which will be discussed in sections 3.2 through 3.5.

This chapter contains several definitions and results that are analogues in rigid cohomology to well-known results about the Betti cohomology of complex-analytical geometric objects. As references for the complex theory we cite [Sai71], [Ste77a], [Ste77b], [Dol82] and [Dim90b]. As should be expected, the main theorem 3.1.11 can also be seen as an analogue to a result over $\mathbb{C}$. See proposition 3.4.3. Although the results in this chapter are to be expected by analogy, the proofs for rigid cohomology are of course more difficult. In fact, we have found that some relevant results over $\mathbb{C}$ cannot be easily translated to rigid cohomology. See section 3.4 for details.

From now on we assume that our base field k is finite, say $k = \mathbb{F}_{p^s}$. As usual we fix a Frobenius $x \mapsto x^q$ $(q = p^r)$ on k together with a lift σ to K.

Throughout this chapter we use the notation $H^\bullet$ as an abbreviation for $H^\bullet_{rig}$. We will also make use of the basic theory of weighted projective spaces, which is covered in [Dol82].

3.1 Weighted homogeneous hypersurface singularities

3.1.1 Definitions and notations

In this paragraph we give the precise definition of a weighted homogeneous hypersurface singularity. We also formulate some additional assumptions that we will use throughout this chapter.

We start by discussing the notion of a hypersurface singularity.

Definition 3.1.1. A singular point $x \in X$ is said to be a *hypersurface singularity* if there exists an $n \geq 2$ and a square-free polynomial $g \in k[x_1, \ldots, x_n]$ satisfying $g(\underline{0}) = 0$ such that (X, x) is contact equivalent to $(Y, 0)$, with $Y = Z_{\mathbb{A}^n_k}(g)$. The polynomial g is called a *local equation* or *normal form* for the singularity $x \in X$. A hypersurface singularity is called *isolated* if the Tjurina number

$$\tau(g) = \dim_k \frac{k[[x_1, \ldots, x_n]]}{(g, \partial_1 g, \ldots, \partial_n g)}$$

is finite.

This definition is slightly restrictive in the sense that it implies that the singularity $x \in X$ is a rational point. This is not a problem for our applications. The condition that the local equation g is square-free ensures that the scheme $Y = Z_{\mathbb{A}^n_k}(g)$ is reduced. Definition 3.1.1 also excludes smooth points from being hypersurface singularities. Therefore the local equation g must have multiplicity ≥ 2, i.e. every monomial must have degree at least 2. It can be verified that for an isolated hypersurface singularity the singular locus $Y_{sing} \subset Y$ is zero-dimensional at the origin.

The name *hypersurface singularity* comes from the fact that $Y = Z_{\mathbb{A}^n_k}(g)$ is an affine hypersurface. The scheme X from definition 3.1.1 on the other hand is an abstract scheme; we make no assumptions about the existence of embeddings. However, if X is an affine or projective hypersurface then all of its singular points are obviously hypersurface singularities.

Observe that the local equation of a hypersurface singularity need not be unique. If $Y' = Z_{\mathbb{A}^n_k}(g')$ and $Y = Z_{\mathbb{A}^n_k}(g)$ for $g', g \in k[x_1, \ldots, x_n]$ then it may be possible that $(Y', 0) \sim_c (Y, 0)$. For hypersurface singularities that are given by a local equation the notion of contact equivalence becomes more concrete. Indeed, for Y' and Y as above it is easy to see that

$$\widehat{\mathcal{O}}_{Y,0} \cong \frac{k[[x_1, \ldots, x_n]]}{(g)} \quad \text{and} \quad \widehat{\mathcal{O}}_{Y',0} \cong \frac{k[[x_1, \ldots, x_n]]}{(g')}.$$

The singularities $0 \in Y'$ and $0 \in Y$ are then contact equivalent if and only if there exists an automorphism φ of $k[[x_1, \ldots, x_n]]$ such that $\varphi(g) = u \cdot g'$ for $u \in k[[x_1, \ldots, x_n]]$ a unit. We also write this as $g' \sim_c g$. This is the definition of contact equivalence that is used in [GK90] and in later papers by Greuel et al.

We proceed by giving the definition of a weighted homogeneous singularity.

Definition 3.1.2. A singular point $x \in X$ is called a *weighted homogeneous hypersurface singularity* or simply a *weighted homogeneous singularity* if it is an isolated hypersurface singularity having a local equation $g \in k[x_1, \ldots, x_n]$ that is weighted homogeneous with respect to weights $\underline{w} = (w_1, \ldots, w_n)$.

In this thesis we use the convention that the weights w_i are integers. We say that a polynomial $g \in k[x_1, \ldots, x_n]$ is *weighted homogeneous of degree d* (or *has weighted degree d*) with respect to the weights $\underline{w}$ if each term of g is of the form $c \cdot x_1^{a_1} \ldots x_n^{a_n}$ with $w_1 a_1 + \ldots + w_n a_n = d$. We also write this as $\deg_{\underline{w}}(g) = d$. Some authors refer to this property as being *quasi-homogeneous*.

Before we continue we briefly consider the situation of an affine or projective hypersurface $X = Z(F)$. It is important to realize that a singularity $x \in X$ may be weighted homogeneous even if the obvious local equations coming from F are not weighted homogeneous. According to definition 3.1.2 a weighted homogeneous singularity $x \in X$ is *contact equivalent to* a weighted homogeneous form, therefore the weighted homogeneous normal form g may only appear after carrying out a suitable change of local coordinates. We illustrate this in example 3.1.3 below.

Example 3.1.3. Consider the projective hypersurface $X = Z(F) \subset \mathbb{P}^4_k$ that is given by the homogeneous equation

$$F(x_0, \ldots, x_4) = \sum_{i=0}^{4} x_i^5 \; - \; 5 \cdot \prod_{i=0}^{4} x_i.$$

This hypersurface is known as *Schoen's quintic.* We wish to show that every singular point on X is weighted homogeneous.

We will make the additional assumption that there exists a primitive 5^{th} root of unity $\zeta \in k$. Under this assumption it is easy to verify that the singular locus of X is given by

$$\left\{ (\zeta^{a_0} : \zeta^{a_1} : \; \ldots : \zeta^{a_4}) \mid \sum_{i=0}^{4} a_i \equiv 0 \mod 5 \right\}.$$

For each of these singular points there is an automorphism of X that maps it to $P = (1 \colon 1 \colon 1 \colon 1 \colon 1)$. Therefore it suffices to study the singularity $P \in X$.

Let us now introduce affine coordinates $y_i = \frac{x_i}{x_0} - 1$ for $i = 1, \ldots, 4$. That is, we limit ourselves to the affine chart $x_0 \neq 0$. This transformation gives us a local equation

$$1 + \sum_{i=1}^{4} (y_i + 1)^5 \; - \; 5 \cdot \prod_{i=1}^{4} (y_i + 1) = 0 \tag{3.1.1}$$

for the hypersurface singularity $P \in X$. This local equation is obviously *not* weighted homogeneous. We will show that $P \in X$ is nevertheless a weighted homogeneous singularity, i.e. that there exists a weighted homogeneous local equation.

First consider the change of coordinates (we additionally assume that $\mathrm{char}(k) \neq 2, 3$):

$$\begin{cases} y_1' &= & y_1 - \frac{1}{4}\, y_2 - \frac{1}{4}\, y_3 - \frac{1}{4}\, y_4 \\ y_2' &= & y_2 - \frac{1}{3}\, y_3 - \frac{1}{3}\, y_4 \\ y_3' &= & y_3 - \frac{1}{2}\, y_4 \\ y_4' &= & y_4 \end{cases}$$

This transformation should be seen as an automorphism of $\mathbb{A}^4_k$, given by the Spec of the map

$$\begin{array}{rcl} k[y_1, \ldots, y_4] & \to & k[y_1, \ldots, y_4] \\ y_i & \mapsto & y_i' \end{array}$$

This automorphism transforms the affine hypersurface given by equation (3.1.1) into the hypersurface with the more manageable equation

$$\sum_{i=1}^{4} y_i^2 \left(b_i + c_i \left(\underline{y} \right) \right) \tag{3.1.2}$$

where $b_i \in k^\times$ and the $c_i \left(\underline{y} \right)$ are polynomials without constant terms in the variables $y_1, \ldots, y_4$. Let us assume for simplicity that the b_i are all squares (for the proof of the general case, see paragraph 4.3.2). After another change of coordinates we then reduce to the case where the b_i are all equal to 1.

Let Y' be the affine hypersurface defined by the equation (3.1.2) with $b_i = 1$ for $i = 1, \ldots, 4$. We now claim that $(Y', 0) \sim_c (Y, 0)$ where $Y = Z_{\mathbb{A}^4_k}(g)$ with g defined by the homogeneous equation

$$g = y_1^2 + y_2^2 + y_3^2 + y_4^2. \tag{3.1.3}$$

In order to prove this, we introduce variables $(\underline{u}\,; \underline{v}\,; \underline{w})$ for $\mathbb{A}^{12}_k$ where $\underline{u} = (u_1, \ldots, u_4)$, $\underline{v} = (v_1, \ldots, v_4)$, and $\underline{w} = (w_1, \ldots, w_4)$. Now look at the variety

$$Y'' \colon \begin{cases} w_i^2 = 1 + c_i\,(\underline{v}) \;\text{ for } i = 1, \ldots, 4 \\ u_i\, w_i = v_i\; (1 + c_i\,(\underline{v})) \;\text{ for } i = 1, \ldots, 4 \\ \sum_{i=1}^{4} v_i^2\; (1 + c_i\,(\underline{v})) = 0 \end{cases}$$

in $\mathbb{A}_k^{12}$. We also introduce the notation $Q := (\underline{0}\,;\,\underline{0}\,;\,\underline{1}) \in Y''$. Now let V denote the open subset $\bigcap_{i=1}^4 D\left(1 + c_i\left(\underline{v}\right)\right)$ of Y''. We can then define two morphisms

$$f' \colon V \longrightarrow Y' \colon (\underline{u}\,;\,\underline{v}\,;\,\underline{w}) \mapsto \underline{v}$$

and

$$f \colon V \longrightarrow Y \colon (\underline{u}\,;\,\underline{v}\,;\,\underline{w}) \mapsto \underline{u}.$$

We show that f' and f are étale at Q using the characterization of étaleness in terms of partial derivatives (see for example [Mil80, Corollary 3.16]). For the Jacobian matrix of f' we obtain:

$$\begin{array}{r|cccccc}
 & u_1 & \ldots & u_4 & w_1 & \ldots & w_4 \\
\hline
u_1\,w_1 - v_1\,\left(1 + c_1\left(\underline{v}\right)\right) & w_1 & & & u_1 & & \\
\vdots & & \ddots & & & \ddots & \\
u_4\,w_4 - v_4\,\left(1 + c_4\left(\underline{v}\right)\right) & & & w_4 & & & u_4 \\
w_1^2 - 1 - c_1\left(\underline{v}\right) & & & & 2w_1 & & \\
\vdots & & & & & \ddots & \\
w_4^2 - 1 - c_4\left(\underline{v}\right) & & & & & & 2w_4
\end{array}$$

The determinant of this matrix evaluated at Q is nonzero. It follows that f' is étale at Q. A similar computation shows that f is étale at Q. We now have that $(Y', 0) \sim_c (Y, 0)$ according to proposition 2.1.2.

We have shown that the singular point $P \in X$ admits the homogeneous local equation (3.1.3). This proves the claim that all the singular points on X are weighted homogeneous.

When we consider a weighted homogeneous singularity we will usually start from a weighted homogeneous normal form g. That is, we take $Y = Z_{\mathbb{A}_k^n}(g)$ and we consider the local cohomology $H^{\bullet}_{\{0\}}(Y)$. To justify this point of view we have to rely on theorem 2.1.1. In the example above we would take $Y = Z_{\mathbb{A}_k^4}(g)$ with $g = x_1^2 + x_2^2 + x_3^2 + x_4^2$. However, we are ultimately interested in the local cohomology space $H^{\bullet}_{\{P\}}(X)$. Theorem 2.1.1 guarantees that both cohomology spaces are the same. In example 3.1.3 the situation is so simple that the Frobenius action on the local cohomology space $H^{\bullet}_{\{0\}}(Y)$ can be computed exactly. We will do this in paragraph 4.3.2.

The idea of passing to a weighted homogeneous form g and applying theorem 2.1.1 is in fact a key idea for proving the main result of this chapter (theorem 3.1.11). Indeed, we will use the form g to define a number of objects in definition 3.1.5 below. All of the proof of theorem 3.1.11 will consist of manipulations of these objects. This is obviously different from the analytical approach to weighted homogeneous singularities that is given in [Sai71]. Indeed, when the base field is $\mathbb{C}$ one usually works in a small complex neighbourhood of the singularity. With this approach there is no need to explicitly change the local coordinates to obtain a weighted homogeneous local equation. However, in our setting we need to immediately change to a weighted homogeneous local form g because the Zariski topology is too coarse to work locally as in [Sai71]. By the results in paragraph 2.1.1 one might also say that we are working étale-locally rather than Zariski-locally. In any case we are using theorem 2.1.1 in an essential way to ensure that the local cohomology doesn't change under the initial transformation.

We now fix a weighted homogeneous normal form g and we discuss the objects that can be defined from it. We start with an auxiliary definition.

Definition 3.1.4. For a polynomial $g \in k[x_1, \ldots, x_n]$ and a set of weights $\underline{w} = (w_1, \ldots, w_n)$ we will write $\widetilde{g} := g(x_1^{w_1}, \ldots, x_n^{w_n})$.

By inspecting the terms of g and $\widetilde{g}$ it is easy to see that g is weighted homogeneous of degree d w.r.t. the given weights if and only if $\widetilde{g}$ is homogeneous of degree d.

Definition 3.1.5. Consider a weighted homogeneous singularity $Y = Z_{\mathbb{A}_k^n}(g)$ with $g \in k[x_1, \ldots, x_n]$ a local equation that is weighted homogeneous of degree d w.r.t. a set of weights $\underline{w} = (w_1, \ldots, w_n)$. We use the normal form g to define the following objects.

- The weighted projective hypersurface $S_\infty = Z(g) \subset \mathbb{P}_k(\underline{w})$.
- The weighted projective hypersurface $S = Z(g - x_0^d) \subset \mathbb{P}_k(1, \underline{w})$ (the additional variable x_0 gets weight 1).
- The projective hypersurface $\widetilde{S}_\infty = Z(\widetilde{g}) \subset \mathbb{P}_k^{n-1}$.
- The projective hypersurface $\widetilde{S} = Z(\widetilde{g} - x_0^d) \subset \mathbb{P}_k^n$.

From this point on we will freely use the notations introduced above. We also use the following convention: after fixing a value for the parameter $n \geq 2$ we use $x_0, x_1, \ldots, x_n$ as coordinates for $\mathbb{P}_k(1, \underline{w})$ (resp. for $\mathbb{P}_k^n$) and $x_1, \ldots, x_n$ as coordinates for $\mathbb{P}_k(\underline{w})$ (resp. for $\mathbb{P}_k^{n-1}$). In this way S_∞ (resp. $\widetilde{S}_\infty$) can be thought of as the space at infinity of S (resp. of $\widetilde{S}$), as the notation suggests.

It follows from our earlier definitions that the hypersurface S_∞ is quasi-smooth:

Proposition 3.1.6. *Let $g \in k[x_1, \ldots, x_n]$ be a weighted homogeneous polynomial of degree d w.r.t. weights $\underline{w}$ such that $Y = Z_{\mathbb{A}_k^n}(g)$ is an* isolated *hypersurface singularity. Then $Y \setminus \{0\}$ is smooth.*

Proof. The Tjurina algebra

$$\frac{k[[x_1, \ldots, x_n]]}{(g, \partial_1 g, \ldots, \partial_n g)}$$

is finite-dimensional over k by assumption, therefore its Krull dimension must be zero. But the Tjurina algebra is the completion of the local ring $\mathcal{O}_{Y_{sing},0}$, therefore the singular locus Y_{sing} is zero-dimensional at the origin.

Since k is perfect we may base-change to $\overline{k}$ and apply the Jacobian criterion [Liu02, Theorem 4.2.19] to $Y_{\overline{k}} \setminus \{0\}$. Assume that $P = (a_1, \ldots, a_n) \in (\overline{k})^n$ is a singular closed point on $Y_{\overline{k}} \setminus \{0\}$. The i-th partial derivative $\partial_i g$ is weighted homogeneous of degree $d - w_i$ and therefore we have an identity

$$\partial_i g(\lambda^{w_1} a_1, \ldots, \lambda^{w_n} a_n) = \lambda^{d-w_i}\, \partial_i g(a_1, \ldots, a_n) = 0$$

for every $\lambda \in \overline{k}$. In other words, the singular locus of $Y_{\overline{k}}$ contains the algebraic curve

$$\left\{(\lambda^{w_1} a_1, \ldots, \lambda^{w_n} a_n) \mid \lambda \in \overline{k}\right\}.$$

We find that the singular locus Y_{sing} has dimension ≥ 1 at the origin. This is a contradiction with the assumption that the singularity defined by g is isolated. $\square$

From now on we will make some additional assumptions about our singularity $Y = Z_{\mathbb{A}^n_k}(g)$.

Definition 3.1.7. Throughout the remainder of this thesis we will assume that the objects from definition 3.1.5 satisfy the following conditions.

i) The weighted degree $d = \deg_{\underline{w}}(g)$ is not divisible by the characteristic of k.

ii) The weights w_i are not divisible by the characteristic of k.

iii) The hypersurface $\widetilde{S}_\infty$ is smooth.

To end this paragraph we give a few comments about the assumptions above. First note that assumptions i) and iii) imply that the hypersurface $\widetilde{S}$ is also smooth. This can be seen by using the Jacobian criterion for smoothness. Indeed, since the ground field k is perfect the smoothness of $\widetilde{S}$ is equivalent to the smoothness of the scheme $Z(\widetilde{g} - x_0^d) \subset \mathbb{P}^n_{\overline{k}}$. Now take a closed point $(a_0, a_1, \ldots, a_n) \in (\overline{k})^{n+1}$ that is a common zero of all the partial derivatives of $\widetilde{g} - x_0^d$. The Jacobian criterion applied to $\widetilde{S}_\infty$ together with assumption iii) implies that $a_1 = \ldots = a_n = 0$. Condition i) ensures that the partial derivative of $\widetilde{g} - x_0^d$ w.r.t. x_0 doesn't vanish, which forces $a_0 = 0$. It follows that $\widetilde{S}$ is indeed smooth. In a similar way one can show that S is quasi-smooth.

In the theory of weighted projective hypersurfaces over $\mathbb{C}$ the usual assumption is that the hypersurfaces under consideration are quasi-smooth. This is for example the case in [Ste77b]. We should emphasize that our assumption iii) is much stronger. Consider for example the following polynomial, which describes a singularity of type D_j ($j \geq 4$).

$$g = x_1^2 + \ldots + x_{n-2}^2 + x_{n-1}^2 x_n + x_n^{j-1}. \tag{3.1.4}$$

It is easy to verify that the associated weighted projective hypersurface S_∞ is quasi-smooth. However, the projective hypersurface $\widetilde{S}_\infty$ has a singularity at the point $(0\colon \ldots : 0\colon 1\colon 0)$.

We see that point iii) is by far the most restrictive assumption in definition 3.1.7. Because of this assumption we have to avoid certain weighted homogeneous singularities that are adequately covered by the theory of Betti cohomology over $\mathbb{C}$. The problem is that the classical proofs over $\mathbb{C}$ rely on some facts that seem to have no obvious analogues in rigid cohomology. We can work around this difficulty if we assume that $\widetilde{S}_\infty$ is smooth. More explanations can be found in section 3.4. Note however that the partial results that we will prove in sections 3.2 and 3.3 do not rely on the assumption that $\widetilde{S}_\infty$ is smooth.

As a positive example, consider a singularity of type A_j, whose normal form is given by

$$g = x_1^2 + \ldots + x_{n-1}^2 + x_n^{j+1}.$$

For j odd we take the weights $\underline{w} = (m, \ldots, m, 1)$ with $m = \frac{j+1}{2}$. For j even we have $\underline{w} = (m, \ldots, m, 2)$ with $m = j + 1$. In either case the conditions of definition 3.1.7 are satisfied if $\mathrm{char}(k) \nmid 2m$. In particular we see that the singularities that we considered in example 3.1.3 satisfy these conditions.

Also note that condition iii) is satisfied in the case of *homogeneous* singularities, i.e. when $g = \widetilde{g}$. This follows immediately from proposition 3.1.6.

Unfortunately we do not know if definition 3.1.7 is invariant under contact equivalence. This is problematic because the local equation of a weighted homogeneous singularity is not unique, meaning that there can be two different weighted homogeneous normal forms g and g' such that $g \sim_c g'$. Take for example $k[x_1, x_2]$, with k a field of characteristic $\neq 2$ that has a square root of 2, and consider the forms

$$g = x_1^2 - x_2^4 \quad \text{and} \quad g' = x_1^2 + x_1 \cdot x_2^2.$$

It is easy to see that $g \sim_c g'$ although $g \neq g'$. We showed before that an equation of the form (3.1.4) doesn't satisfy the conditions of definition 3.1.7. Strictly speaking this does not imply that a singularity of *type* D_j doesn't satisfy these conditions. In principle it is possible that there exists another weighted homogeneous normal form that *does* satisfy the conditions of definition 3.1.7.

This problem is deeper than it may seem at first. In our context we do not even know if conditions i) and ii) from definition 3.1.7 are invariant under contact equivalence. But it is interesting to note that this question has been solved over $\mathbb{C}$. Indeed, fix a prime number p, write $d = \deg_{\underline{w}}(g)$ and also assume that $\gcd(w_1, \ldots, w_n) = 1$. Now consider the reduced fractions $\frac{w_i}{d} = \frac{a_i}{b_i}$ for $i = 1, \ldots, n$. Then the first two points of definition 3.1.7 (with respect to the chosen prime p) can be reformulated as:

$$\text{For every index } i,\ p \nmid a_i \text{ and } p \nmid b_i.$$

It has been shown in [Sai71, Lemma 4.3] that over the base field $\mathbb{C}$, the numbers $\frac{w_i}{d}$ do not change if g is replaced by a weighted homogeneous form $g' \sim_c g$. At least, this is the case if one only considers forms whose weights satisfy $\frac{w_i}{d} \leq \frac{1}{2}$ for every i. Unfortunately, the proof heavily relies on analytic methods. Also see [Sae98] for a topological approach. We are not aware of any proof that works over a base field of positive characteristic.

3.1.2 Group actions, base field extensions and Frobenius

Before we can state the main result of this chapter we need to settle some subtle points about certain group actions and their interaction with rigid cohomology.

Recall from [Dol82] that the weighted projective space $\mathbb{P}_k(1, \underline{w})$ can be constructed as a quotient of $\mathbb{P}^n_k$ by a certain group scheme. After base-changing to the algebraic closure $\overline{k}$ this comes down to the same thing as quotienting by an (ordinary) finite group acting on $\mathbb{P}^n_{\overline{k}}$ (at least if the weights are not divisible by the characteristic of k). The latter description is good for our applications because by functoriality we also obtain a group action on rigid cohomology. But over k itself it is not immediately clear how an action of a group scheme interacts with rigid cohomology.

To solve this problem we give a series of definitions that we will use throughout this chapter and chapter 4. With these definitions we avoid the use of group schemes, although intuitively nothing changes.

Definition 3.1.8. Assume that the ground field k contains unit roots $\zeta_{w_1}, \ldots, \zeta_{w_n}$ of order $w_1, \ldots, w_n$ respectively. That is: $(\zeta_{w_i})^{w_i} = 1$ but $\zeta_{w_i}^a \neq 1$ for $0 < a < w_i$. Then consider the group

$$G(\underline{w}) := \bigoplus_{i=1}^{n} \langle \zeta_{w_i} \rangle.$$

We define a right[1] action of $G(\underline{w})$ on $\mathbb{P}^n_k$ as follows. For an element $h = (\zeta_{w_1}^{a_1}, \ldots, \zeta_{w_n}^{a_n}) \in G(\underline{w})$ we define $\phi_h \in \operatorname{Aut}(\mathbb{P}^n_k)$ as the Proj of the graded

[1] We follow the convention from Exposé V of [SGA1] that a group of automorphisms $G \subset \operatorname{Aut}(X)$ acts on X from the right.

algebra morphism

$$\begin{array}{rcl} k[x_0, x_1, \dots, x_n] & \to & k[x_0, x_1, \dots, x_n] \\ x_i & \mapsto & \zeta_{w_i}^{a_i} x_i \ \text{ for } i \neq 0 \\ x_0 & \mapsto & x_0 \end{array}$$

The action of $G(\underline{w})$ on $\mathbb{P}^n_k$ restricts to an action on $\mathbb{P}^{n-1}_k = Z_{\mathbb{P}^n_k}(x_0)$. The equations $\widetilde{g} - x_0^d$ and $\widetilde{g}$ are obviously invariant under $G(\underline{w})$ and therefore the schemes $\widetilde{S}$, $\widetilde{S}_\infty$, $\mathbb{P}^n_k \setminus \widetilde{S}$, $\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty$ and $\widetilde{S} \setminus \widetilde{S}_\infty$ are also equipped with an action of $G(\underline{w})$. Moreover, all the inclusions between these schemes (such as $\widetilde{S}_\infty \hookrightarrow \mathbb{P}^{n-1}_k$ or $\widetilde{S} \setminus \widetilde{S}_\infty \hookrightarrow \widetilde{S}$) are $G(\underline{w})$-equivariant.

By functoriality we also have a $G(\underline{w})$-action on the cohomology spaces of all the schemes that we mentioned above. As a result we may consider the $G(\underline{w})$-invariant parts $H^\bullet(\underline{\quad})^{G(\underline{w})}$ of these cohomology spaces.

Since we assumed that k contains the unit roots $\zeta_{w_1}, \dots, \zeta_{w_n}$ it is obvious that the quotient of $\mathbb{P}^n_k$ by $G(\underline{w})$ is isomorphic to $\mathbb{P}_k(1, \underline{w})$. Likewise we have $\mathbb{P}^{n-1}_k / G(\underline{w}) = \mathbb{P}_k(\underline{w})$, $\widetilde{S}/G(\underline{w}) = S$, $\widetilde{S}_\infty / G(\underline{w}) = S_\infty$ and $(\widetilde{S} \setminus \widetilde{S}_\infty)/G(\underline{w}) = S \setminus S_\infty$.

The biggest restriction in the definitions above is the assumption that $\zeta_{w_1}, \dots, \zeta_{w_n} \in k$. Note however that there exists a finite extension $k' \supset k$ containing all these roots. This follows from our earlier assumption that $\mathrm{char}(k) \nmid w_i$ for all i. Indeed, under this assumption all the polynomials $T^{w_i} - 1 \in k[T]$ are separable, meaning that there exist w_i distinct elements $\alpha \in \overline{k}$ such that $\alpha^{w_i} = 1$. A generator for the group of all such roots must have order w_i.

In the remainder of this paragraph we will argue that under the assumptions of definition 3.1.7 we can still make sense of the $G(\underline{w})$-invariant cohomology spaces $H^\bullet(\underline{\quad})^{G(\underline{w})}$ for a base field k that may not contain all of the roots $\zeta_{w_1}, \dots, \zeta_{w_n}$. For this we use the following result by Tsuzuki.

Proposition 3.1.9. *Consider a base field k with Frobenius $x \mapsto x^q$. Let $K = \mathrm{Frac}\,\mathcal{V}$ be a complete ultrametric field such that $\mathcal{V}/(\pi) \cong k$. As usual we assume that the Frobenius on k admits a lift σ to K. Now consider an algebraic extension $k' \supset k$ together with an extension $K' = \mathrm{Frac}\,\mathcal{V}'$ of K such that $\mathcal{V}'/(\pi') \cong k'$. Also assume that K' admits a lift σ' of the q-power Frobenius on k', which moreover satisfies $\sigma'_{|K} = \sigma$. Finally consider a k-scheme X, whose extension to k' we denote by $X_{k'}$. Then there is a canonical Frobenius-equivariant isomorphism*

$$H^\bullet_{rig}(X) \otimes_{\sigma'} K' \xrightarrow{\sim} H^\bullet_{rig}(X_{k'}). \tag{3.1.5}$$

Proof. See [Tsu99, Theorem 6.1.1] in the case where X is smooth. The smoothness condition can be removed by using cohomological descent. The proof of this last statement is similar to (and easier than) the proof of proposition 3.2.2 in the next section. $\square$

From now on we fix an algebraic extension k' of k that contains the roots $\zeta_{w_1}, \ldots, \zeta_{w_n}$ and we assume that there exists a lift $\sigma'\colon K' \to K'$ as in the proposition above. Note that when k and k' are algebraic over $\mathbb{F}_p$ and if we take K and K' to be the completions of unramified algebraic extensions of $\mathbb{Q}_p$ then the lift σ' exists and is unique.

Now let X denote one of the k-schemes $\mathbb{P}^n_k$, $\mathbb{P}^{n-1}_k$, $\widetilde{S}$, $\widetilde{S}_\infty$, $\mathbb{P}^n_k \setminus \widetilde{S}$, $\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty$ or $\widetilde{S} \setminus \widetilde{S}_\infty$. Then $X_{k'}$ is equipped with a $G(\underline{w})$-action and proposition 3.1.9 allows us to define the $G(\underline{w})$-invariant subspace of $H^\bullet(X)$.

Definition 3.1.10. With X as above and $i \geq 0$ we define $H^i(X)^{G(\underline{w})}$ to be the intersection with $H^i(X) \subset H^i(X) \otimes K'$ of the inverse image of $H^i(X_{k'})^{G(\underline{w})}$ under the canonical map (3.1.5).

With this definition we have all the necessary material to formulate our main theorem 3.1.11 in the next paragraph. We end this paragraph with a remark that we will use throughout the proof of theorem 3.1.11.

The fact that the isomorphism of proposition 3.1.9 is canonical means that if $f\colon X \to Y$ is a morphism of k-schemes then we have a commutative diagram as follows:

$$\begin{array}{ccc} H^\bullet(Y) \otimes_{\sigma'} K' & \xrightarrow{\sim} & H^\bullet(Y_{k'}) \\ {\scriptstyle H^\bullet(f) \otimes \mathrm{Id}_{K'}} \downarrow & & \downarrow {\scriptstyle H^\bullet(f_{k'})} \\ H^\bullet(X) \otimes_{\sigma'} K' & \xrightarrow{\sim} & H^\bullet(X_{k'}) \end{array}$$

This means that the map $H^\bullet(f)$ may be recovered from $H^\bullet(f_{k'})$ by restricting the source and target to $H^\bullet(Y) \subset H^\bullet(Y_{k'})$ resp. to $H^\bullet(X) \subset H^\bullet(X_{k'})$.

Another fact is that the Gysin isomorphism and the Künneth formula are compatible with base field extensions. This can be proved from proposition 3.1.9 and its counterpart for cohomology with compact supports (see [Tsu99, Theorem 6.1.2]).

We can combine these facts to make the following observation. Assume that we are working with k'-schemes *that are defined over* k and that we construct a map using only the following operations:

- Applying $H^\bullet$ to a morphism $f\colon X_{k'} \to Y_{k'}$ *that is defined over* k.
- Using the long exact sequence with supports (1.2.18).
- Using the Gysin isomorphism and the Künneth formula.

Then we can always bring the situation back down to k. We will implicitly use this observation throughout the rest of this chapter.

3.1.3 Statement of results

Theorem 3.1.11. *Let $g \in k[x_1, \dots, x_n]$ be a weighted homogeneous form w.r.t. a tuple of weights $\underline{w} = (w_1, \dots, w_n)$. Assume that the singularity $0 \in Y = Z_{\mathbb{A}^n_k}(g)$ satisfies all the assumptions of definition 3.1.7. Also assume that $n \geq 3$. Then the following properties hold:*

i) Using the notation of paragraphs 3.1.1 and 3.1.2, there is a Frobenius-equivariant isomorphism

$$H^n_{\{0\}}(Y) \xrightarrow{\sim} H^{n-1}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}.$$

ii) There is a Frobenius-equivariant isomorphism

$$H^{n-1}_{\{0\}}(Y)(-1) \xrightarrow{\sim} H^n_{\{0\}}(Y).$$

iii) We have $H^i_{\{0\}}(Y) = 0$ for $i \notin \{n-1, n, 2n-2\}$. The space $H^{2n-2}_{\{0\}}(Y)$ is one-dimensional and its Frobenius action is given by $q^{n-1}\sigma$.

Also see table 3.1 below.

i	$\dim H^i_{\{0\}}(Y)$
$0 \leq i \leq n-2$	0
$n-1$	N
n	N
$n+1 \leq i \leq 2n-3$	0
$2n-2$	1
$> 2n-2$	0

Table 3.1: Betti table of a weighted homogeneous hypersurface singularity

Remark 3.1.12. The first statement of the theorem can be used to effectively approximate the Frobenius action on the local cohomology $H^n_{\{0\}}(Y)$, using a modification of the algorithm from [AKR11]. This will be the main result of chapter 4.

Remark 3.1.13. Theorem 3.1.11 has a straightforward extension to the case $n = 2$. See remark 3.5.11 for details.

3.2 Expressing local cohomology in terms of Monsky-Washnitzer cohomology

The goal of this section is to express the local cohomology $H^i_{\{0\}}(Y)$ of a weighted homogeneous hypersurface singularity $Y = Z_{\mathbb{A}^n_k}(g)$ in terms of Monsky-Washnitzer cohomology. More precisely, we prove the following proposition.

Proposition 3.2.1. *Consider as before an isolated weighted homogeneous hypersurface singularity $Y = Z_{\mathbb{A}^n_k}(g)$, given by a weighted homogeneous local equation g. We also assume that $n \geq 3$. Then the following properties hold:*

i) We have $H^i_{\{0\}}(Y) = 0$ for $i = 0, 1$.

ii) There is a Frobenius-equivariant isomorphism

$$H^i_{\{0\}}(Y) \xrightarrow{\sim} H^i(\mathbb{A}^n_k \setminus Y)(+1)$$

for $2 \leq i \leq 2n - 3$.

iii) The space $H^{2n-2}_{\{0\}}(Y)$ is one-dimensional and its Frobenius action is given by $q^{n-1}\sigma$.

iv) We have $H^i_{\{0\}}(Y) = 0$ for $i > 2n - 2$.

The scheme $\mathbb{A}^n_k \setminus Y$ is smooth and affine, so by the comparison theorem with Monsky-Washnitzer cohomology we conclude that the local cohomology can be expressed in terms of the spaces $H^i_{MW}(\mathbb{A}^n_k \setminus Y)$.

Proposition 3.2.1 should not come as a surprise, since the proof for Betti cohomology with coefficients in $\mathbb{C}$ is a simple exercise. And indeed, the proof for rigid cohomology is not particularly difficult. The main technical difficulty is that we will need to use cohomological descent for the proof of proposition 3.2.2 below. We have not found any elementary way to prove this proposition, even though the stated property is intuitively clear.

Also note that the statement of proposition 3.2.1 is generally false if one chooses another local equation for the singularity $0 \in Y$. As we explained in paragraph 3.1.1, we have to assume from the start that g is weighted homogeneous. By doing so we are heavily relying on theorem 2.1.1.

3.2.1 Weighted homogeneous normal forms are acyclic

The first step towards the proof of proposition 3.2.1 is to show that the affine scheme $Y = Z_{\mathbb{A}^n_k}(g)$ is acyclic, i.e. that $H^i(Y) = 0$ for every $i > 0$. This result can be compared with point (b) of the theorem of [Sai71]. The main difference is that we work with a weighted homogeneous local equation inside of $\mathbb{A}^n_k$, rather than with an arbitrary local equation in a small (complex) neighbourhood of the origin. We start with a technical lemma.

Proposition 3.2.2. *Let Y be a scheme of finite type and separated over k. Then for every $i \geq 0$ the rigid cohomology spaces $H^i(Y)$ and $H^i(Y \times \mathbb{A}^1_k)$ have equal dimension.*

Proof. If Y is smooth then this easily follows from the Künneth formula. We use de Jong's alteration theorem and cohomological descent to reduce to the smooth case. Indeed, the alteration theorem can be used to show that there exists a proper hypercover $a\colon X_\bullet \to Y$ such that X_n is smooth for every n. See the introduction of [dJ96] or [Con03, Theorem 4.7] for details.

We now use [ZB14, Theorem 1.1], which states that proper hypercovers are of cohomological descent w.r.t. finitely presented modules. This means that in the derived category there is an isomorphism between $\mathcal{O}^\dagger_Y$ and the equalizer of the arrows

$$\mathbb{R}(a_0)_*\mathcal{O}^\dagger_{X_0} \rightrightarrows \mathbb{R}(a_1)_*\mathcal{O}^\dagger_{X_1} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

We may also take the product of a with the constant augmented simplicial complex on $\mathbb{A}^1_k$. We write this as $a'\colon X'_\bullet \to Y \times \mathbb{A}^1_k$ where $X'_n = X_n \times \mathbb{A}^1_k$. According to [Con03, Lemma 4.6] a' is again a proper hypercover. If pr_i, $i = 1, 2$ denote the projections from X'_n then we have of course an identification of (classical) overconvergent isocrystals:

$$\mathcal{O}_{X'_n/K} = pr_1^*\mathcal{O}_{X_n/K} \otimes_{\mathcal{O}_{X'_n/K}} pr_2^*\mathcal{O}_{\mathbb{A}^1_k/K}. \tag{3.2.1}$$

Now let $p\colon Y \to \operatorname{Spec} k$ resp. $p'\colon Y \times \mathbb{A}^1_k \to \operatorname{Spec} k$ denote the structural morphisms of Y resp. of $Y \times \mathbb{A}^1_k$. Since the X_n and X'_n are smooth and since $\mathbb{A}^1_k$ has no rigid cohomology outside of degree zero we may apply the Künneth formula (see theorem 1.2.9) to the identification (3.2.1). Together with the comparison theorem [LS11, Corollary 4.6.8] this gives a canonical isomorphism

$$\mathbb{R}(p \circ a_n)_*\mathcal{O}^\dagger_{X_n} \xrightarrow{\sim} \mathbb{R}(p' \circ a'_n)_*\mathcal{O}^\dagger_{X'_n}$$

for every $n \geq 0$. We now have a commutative diagram

$$\begin{tikzcd}
\mathbb{R}(p \circ a_0)_*\mathcal{O}^\dagger_{X_0} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \mathbb{R}(p \circ a_1)_*\mathcal{O}^\dagger_{X_1} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\cong"] & \cdots \\
\mathbb{R}(p' \circ a'_0)_*\mathcal{O}^\dagger_{X'_0} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{R}(p' \circ a'_1)_*\mathcal{O}^\dagger_{X'_1} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd}$$

Since $\mathbb{R}p_*$ is a right adjoint it preserves limits, hence theorem [ZB14, Theorem 1.1] gives an isomorphism between the equalizer of the top row and $\mathbb{R}p_*\mathcal{O}^\dagger_Y$. A similar property holds for the equalizer of the bottom row and $\mathbb{R}p'_*\mathcal{O}^\dagger_{Y\times\mathbb{A}^1_k}$. So we find an isomorphism $\mathbb{R}p_*\mathcal{O}^\dagger_Y \xrightarrow{\sim} \mathbb{R}p'_*\mathcal{O}^\dagger_{Y\times\mathbb{A}^1_k}$ in the derived category. By applying the comparison theorem [LS11, Corollary 4.6.8] we obtain an isomorphism $H^\bullet(Y) \xrightarrow{\sim} H^\bullet(Y \times \mathbb{A}^1_k)$. □

The acyclicity of Y now follows from a rather standard argument. Compare for example with paragraph 4 of [BT82].

Proposition 3.2.3. *Let $g \in k[x_1, \ldots x_n]$ be a weighted homogeneous form with weights $w_1, \ldots, w_n$ and write $Y = Z_{\mathbb{A}^n_k}(g)$. Then $H^i(Y) = 0$ for every $i > 0$.*

Proof. Define a map

$$F \colon Y \times \mathbb{A}^1_k \to Y \colon (x_1, \ldots, x_n, t) \mapsto ((1-t)^{w_1} x_1, \ldots, (1-t)^{w_n} x_n).$$

Also consider the sections σ_j for $j = 0, 1$:

$$\sigma_j \colon Y \to Y \times \mathbb{A}^1_k \colon \underline{x} \mapsto (\underline{x}, j).$$

If p denotes the projection onto Y then we have $p \circ \sigma_j = \mathrm{Id}_Y$, so on cohomology we have $\sigma_j^* \circ p^* = \mathrm{Id}_Y^*$. Since we know by proposition 3.2.2 that $H^\bullet(Y)$ and $H^\bullet(Y \times \mathbb{A}^1)$ have equal dimension it follows that the maps σ_j^* are invertible. Also observe that $F \circ \sigma_0$ is the identity on Y and that $F \circ \sigma_1$ maps every point of Y to the origin. So $F \circ \sigma_1$ factors through the one-point space $\{0\}$ and it follows that the map on cohomology $\sigma_1^* \circ F^* \colon H^i(Y) \to H^i(Y)$ is zero for $i > 0$. We now have two commutative diagrams as follows, for any $i > 0$:

$$\begin{tikzcd}
H^i(Y) \arrow[r, "F^*"] \arrow[rr, bend left, "Id_Y^*"] \arrow[rr, bend right, "0"'] & H^i(Y \times \mathbb{A}^1_k) \arrow[r, shift left, "\sigma_0^*"] \arrow[r, shift right, "\sigma_1^*"'] & H^i(Y)
\end{tikzcd}$$

On the one hand we see that F^* is an isomorphism. On the other hand we have that $F^* = 0$. It follows that $H^i(Y) = 0$ for $i > 0$. $\square$

3.2.2 Local cohomology and Monsky-Washnitzer cohomology

With proposition 3.2.3 in place we now proceed with the proof of proposition 3.2.1. Essentially this boils down to an application of the Gysin sequence (1.2.20), but we need a few lemmas first.

Proposition 3.2.4.

i) For every $i \geq 2$ there is a Frobenius-equivariant isomorphism

$$H^{i-1}(Y \setminus \{0\}) \xrightarrow{\sim} H^i_{\{0\}}(Y).$$

ii) If the canonical map $H^0(Y) \to H^0(Y \setminus \{0\})$ is nonzero then $H^i_{\{0\}}(Y) = 0$ for $i = 0, 1$.

Proof. Consider the long exact sequence with supports

$$\ldots \longrightarrow H^i_{\{0\}}(Y) \longrightarrow H^i(Y) \longrightarrow H^i(Y \setminus \{0\}) \longrightarrow \ldots$$

It follows from proposition 3.2.3 that the boundary maps give an isomorphism $H^{i-1}(Y \setminus \{0\}) \cong H^i_{\{0\}}(Y)$ for $i \geq 2$. Moreover, the beginning of the sequence is given by

$$0 \longrightarrow H^0_{\{0\}}(Y) \longrightarrow H^0(Y) \longrightarrow H^0(Y \setminus \{0\}) \longrightarrow H^1_{\{0\}}(Y) \longrightarrow 0$$

and since $\dim H^0(Y) = \dim H^0(Y \setminus \{0\}) = 1$ this proves the second claim. $\square$

It is not difficult to see that the canonical map $H^0(Y) \to H^0(Y \setminus \{0\})$ is indeed nonzero. In fact, this is a special case of a very general property, which we prove below.

Proposition 3.2.5. *Let Y be a scheme of finite type and separated over k. Consider the inclusion $U \hookrightarrow Y$ of a dense open subset. Then the canonical map $H^0(Y) \to H^0(U)$ is an isomorphism.*

Proof. Choose a proper hypercover $X_\bullet \to Y$ as in the proof of proposition 3.2.2. According to [Con03, Lemma 4.6] we can pull back $X_\bullet$ along $U \hookrightarrow Y$ to obtain a proper hypercover $X'_\bullet \to U$. By construction, each map $X'_n \to X_n$ is the inclusion of a dense open subset. In this way we reduce to the case where Y is smooth.

Obviously we may also assume that Y is connected, hence of pure dimension. Denote this dimension by d. Then by [LS07, Proposition 8.2.21] we have an isomorphism $H^{2d}_{rig,c}(U) \xrightarrow{\sim} H^{2d}_{rig,c}(Y)$. The proposition follows by applying Poincaré duality. $\square$

With this we have already proved point i) of proposition 3.2.1. Point iv) was only mentioned for completeness, as this is a general fact of rigid cohomology. To see that this property is true, we first use cohomological descent to reduce to the smooth case. Then choose a cover by smooth affine opens and apply the Čech spectral sequence from [LS07, Proposition 8.2.17]. We now proceed with the proof of points ii) and iii).

Proof of Proposition 3.2.1. Proposition 3.1.6 implies that we may apply the Gysin isomorphism to the pair $Y \setminus \{0\} \subset \mathbb{A}^n_k \setminus \{0\}$. So consider the Gysin sequences

$$\ldots \longrightarrow H^i(\mathbb{A}^n_k \setminus Y) \longrightarrow H^{i-1}(Y \setminus \{0\})(-1) \longrightarrow H^{i+1}(\mathbb{A}^n_k \setminus \{0\}) \longrightarrow \ldots \tag{3.2.2}$$

and

$$\ldots \longrightarrow H^i(\mathbb{A}^n_k \setminus \{0\}) \longrightarrow H^{i-2n+1}(\{0\})(-n) \longrightarrow H^{i+1}(\mathbb{A}^n_k) \longrightarrow \ldots \tag{3.2.3}$$

It follows directly that $H^i(\mathbb{A}^n_k \setminus \{0\}) = 0$ for $1 \leq i \leq 2n-2$. This gives us an isomorphism $H^i(\mathbb{A}^n_k \setminus Y) \cong H^{i-1}(Y \setminus \{0\})(-1)$ for $1 \leq i \leq 2n-3$. Combining this with proposition 3.2.4 we find an isomorphism

$$H^i(\mathbb{A}^n_k \setminus Y)(+1) \cong H^{i-1}(Y \setminus \{0\}) \cong H^i_{\{0\}}(Y)$$

for $2 \leq i \leq 2n-3$. This proves point ii). By looking at the end of the sequence (3.2.2) it follows that we have an isomorphism $H^{2n-3}(Y \setminus \{0\})(-1) \cong H^{2n-1}(\mathbb{A}^n_k \setminus \{0\})$ if $n \geq 3$. To see this you also need to use that $H^i(\mathbb{A}^n_k \setminus Y) = 0$ for $i \in \{2n-2, 2n-1\}$. This follows from the comparison theorem with Monsky-Washnitzer cohomology because $\mathbb{A}^n_k \setminus Y$ is smooth affine of dimension n and $2n-2 > n$. The sequence (3.2.3) also tells us that $H^{2n-1}(\mathbb{A}^n_k \setminus \{0\}) \cong H^0(\{0\})(-n)$. It follows that

$$H^{2n-2}_{\{0\}}(Y) \cong H^{2n-3}(Y \setminus \{0\}) \cong H^{2n-1}(\mathbb{A}^n_k \setminus \{0\})(+1) \cong H^0(\{0\})(1-n).$$

We see that $H^{2n-2}_{\{0\}}(Y)$ is one-dimensional with Frobenius equal to $q^{n-1}\sigma$, which proves point iii). $\square$

To finish this section we prove a simple corollary of proposition 3.2.1.

Corollary 3.2.6. *If $n \geq 4$ then we have that $H^i_{\{0\}}(Y) = 0$ for $n+1 \leq i \leq 2n-3$.*

Proof. Since $\mathbb{A}^n_k \setminus Y$ is smooth affine of dimension n we have that $H^i(\mathbb{A}^n_k \setminus Y) = H^i_{MW}(\mathbb{A}^n_k \setminus Y) = 0$ for $i > n$. The result follows directly from proposition 3.2.1. $\square$

To prove point iii) of theorem 3.1.11 it now suffices to show that $H^i_{\{0\}}(Y) = 0$ for $1 < i < n-1$. This will be done in section 3.5.

3.3 Local cohomology and the affine Milnor fiber

In the previous section we showed that the local cohomology $H^\bullet_{\{0\}}(Y)$ may be identified with the cohomology space $H^\bullet(\mathbb{A}^n_k \setminus Y)(+1)$. The next step is to derive a relation between $H^\bullet(\mathbb{A}^n_k \setminus Y)$ and the monodromy-invariant part of the cohomology of the affine Milnor fiber. The precise statement is given in proposition 3.3.10.

We start by giving the definition of the affine Milnor fiber and its monodromy action in paragraph 3.3.1 below. In the paragraph after that we recall some recent work by Etesse about the behaviour of rigid cohomology w.r.t. étale Galois covers. We apply this to the Monsky-Washnitzer cohomology of the quotient of an affine scheme by a finite group. With this preliminary material the proof of proposition 3.3.10 becomes quite straightforward: we essentially construct a map from the affine Milnor fiber to $\mathbb{A}^n_k \setminus Y$ and we show that it is the quotient map w.r.t. the monodromy action.

3.3.1 The affine Milnor fiber and its monodromy action

In the study of germs of analytic functions over $\mathbb{C}$ there is a classical object called the *Milnor fiber*. Although the Milnor fiber is defined through analytic geometry, it is known that the Milnor fiber of a function germ that is given by a weighted homogeneous polynomial g is diffeomorphic to the zero set of $g-1$. In [Dim92, Definition 3.1.12] this object is called the *affine Milnor fiber*.

The (affine) Milnor fiber plays an important role in the study of weighted homogeneous singularities over $\mathbb{C}$. It is therefore not surprising that we will need an algebraic version of the affine Milnor fiber. See the definition below.

Definition 3.3.1. Consider a weighted homogeneous polynomial g satisfying the assumptions of definition 3.1.7. Then we define the *affine Milnor fiber* to be the affine scheme $S \setminus S_\infty$. If one considers the coordinates $X_i := \frac{x_i}{x_0}$ for $\mathbb{A}^n_k$ then it is clear that the affine Milnor fiber is isomorphic to $Z_{\mathbb{A}^n_k}(g-1)$.

Proposition 3.3.2. *Under our usual assumptions the affine Milnor fiber is smooth.*

Proof. This is easy to see using the Jacobian criterion for smoothness (as in the proof of proposition 3.1.6) together with the Euler relation for the weighted homogeneous polynomial g. □

Remark 3.3.3. Note that the assumptions of definition 3.1.7 imply that the affine Milnor fiber $\widetilde{S} \setminus \widetilde{S}_\infty$ associated to $\widetilde{g}$ is also smooth.

We also give an algebraic definition of the monodromy action on the affine Milnor fiber.

Definition 3.3.4. Let g be a weighted homogeneous polynomial of degree d w.r.t. weights $(w_1, \ldots, w_n)$ satisfying the assumptions of definition 3.1.7. Fix an algebraic extension k' of k that contains a unit root ζ_d of order d. Then we let the cyclic group $\langle \zeta_d \rangle$ act from the right on $\mathbb{P}_{k'}(1, \underline{w})$ by mapping ζ_d to the Proj of the graded algebra morphism

$$\begin{aligned} k'[x_0, x_1, \ldots, x_n] &\to k'[x_0, x_1, \ldots, x_n] \\ x_i &\mapsto \zeta_d^{w_i} x_i \quad \text{for } i \neq 0 \\ x_0 &\mapsto x_0 \end{aligned}$$

It is clear that both $g - x_0^d$ and the space at infinity are invariant under $\langle \zeta_d \rangle$. Therefore we also have an action of $\langle \zeta_d \rangle$ on $S \setminus S_\infty$. We call this the *monodromy action* on the affine Milnor fiber.

This definition is inspired by the monodromy diffeomorphism of a weighted homogeneous germ over $\mathbb{C}$. Indeed, with respect to the coordinates $X_i = \frac{x_i}{x_0}$ the monodromy action on $Z_{\mathbb{A}^n_k}(g-1)$ can be described as the Spec of the algebra morphism

$$\begin{array}{rcl} k'[X_1, \dots, X_n] & \to & k'[X_1, \dots, X_n] \\ X_i & \mapsto & \zeta_d^{w_i} X_i \end{array}$$

This is analogous to the monodromy diffeomorphism, as shown in [Dim92, Example 3.1.19].

It is obvious that $\langle \zeta_d \rangle$ also acts on the schemes S, $\mathbb{P}_{k'}(1, \underline{w}) \setminus S$, $\mathbb{P}_{k'}(\underline{w})$, S_∞ and $\mathbb{P}_{k'}(\underline{w}) \setminus S_\infty$. The action on the last three schemes is trivial. We may also define the $\langle \zeta_d \rangle$-invariant subspaces of the cohomology of the corresponding schemes over k, analogously to definition 3.1.10.

3.3.2 Rigid cohomology and étale Galois covers

Consider a k-scheme X with a right action of a finite group G. We denote such a group action as a homomorphism $\phi\colon G \to \operatorname{Aut}(X)$. Each element $g \in G$ then gives us an automorphism $\phi_g := \phi(g) \in \operatorname{Aut}(X)$. For simplicity we will assume that the action is faithful, which means that the homomorphism ϕ is injective.

Assume moreover that X can be covered by G-stable affine opens. This condition is automatically satisfied for a quasi-projective X. It is shown in exposé V in [SGA1] that for such a scheme X there exists a scheme Y and a morphism $q\colon X \to Y$ satisfying all the properties listed below. Using the terminology of [MFK94], we say that Y is the *geometric quotient* for the G-action on X. The morphism q is called the *quotient map.*

- The quotient map q is finite and surjective. The quotient map is also invariant under G: we have $q \circ \phi_g = q$ for every $g \in G$.
- The group G acts transitively on the geometric fibers of q.
- When viewed as a continuous map, q is precisely the quotient map w.r.t. the Zariski topology on X.
- There is a canonical isomorphism $\mathcal{O}_Y \xrightarrow{\sim} q_*^G(\mathcal{O}_X)$.

It can be shown that the pair (Y, q) is also a *categorical quotient*: if $a\colon X \to Z$ is another morphism such that $a \circ \phi_g = a$ for all $g \in G$, then there is a unique arrow $b\colon Y \to Z$ such that $a = b \circ q$.

The following additional properties are also useful for the remainder of this chapter:

- If $X = \operatorname{Spec} A$ is affine then by functoriality A is equipped with a left G-action. In this case $Y \cong \operatorname{Spec} A^G$ and the quotient map corresponds to the Spec of the inclusion $A^G \hookrightarrow A$.
- If G acts freely on $X_{\overline{k}}$ (obtained by base-changing to the algebraic closure) then q is étale.

The quotient map q induces a map $H^\bullet(q)\colon H^\bullet(Y) \to H^\bullet(X)$ on rigid cohomology. By functoriality $H^\bullet(X)$ is equipped with a left G-action. Since the quotient map is G-equivariant by construction, the map $H^\bullet(q)$ restricts to a map

$$H^\bullet(Y) \to H^\bullet(X)^G.$$

It is natural to ask if this map is an isomorphism. Etesse gave an affirmative answer in the case where Y is smooth and the quotient map is an étale Galois cover.

We say that a morphism $f\colon X \to Y$ is a *finite étale Galois cover* (or simply an *étale Galois cover*) if f is finite, surjective, étale and the Galois group $\mathrm{Aut}(X/Y)$ acts transitively on the geometric fibers of f. See chapter 5 in [Sza09] for a detailed discussion of this type of morphism.

Proposition 3.3.5. *Let $f\colon X \to Y$ be a finite étale Galois cover with group $G = Aut(X/Y)$. Assume moreover that Y is smooth. Then the canonical map*

$$H^i(f)\colon H^i(Y) \to H^i(X)^G$$

is an isomorphism for every $i \geq 0$.

Proof. This is a special case of [Ete08, Théorème IV.4.2]. □

Recall that the morphism f in the proposition above is precisely the quotient map associated to the G-action (at least when X is connected). Conversely, if a quotient map $X \to Y = X/G$ is an étale Galois cover then $G \cong \mathrm{Aut}(X/Y)$. See for instance [Sza09, Proposition 5.3.8].

In this section we will apply proposition 3.3.5 to Monsky-Washnitzer cohomology, i.e. we consider the case where X and Y are smooth affine. We use a slightly different formulation that will be useful later on.

Proposition 3.3.6. *Let $X = \operatorname{Spec} A$ be a smooth affine scheme with an action of a finite group G and assume that the quotient $Y = \operatorname{Spec} A^G$ is again smooth. Then the following properties hold.*

i) The canonical map $H^i(Y) \to H^i(X)^G$ is injective for every i.

ii) If moreover G acts freely on $X_{\overline{k}}$ then this map is an isomorphism.

Proof.

i) This is a special case of [Ber97b, Proposition 3.6]. Note that the flatness assumption in the cited property is automatically satisfied in our setting, thanks to the *miracle flatness property.* This property implies that a finite surjective morphism between smooth schemes is flat. See [Liu02, Remark IV.3.11] for details.

ii) The assumption that G acts freely on $X_{\overline{k}}$ implies that the quotient map $q\colon X \to Y$ is étale. See for example [SGA1, Corollaire V.2.3]. Combining this with the usual properties of quotient maps we see that q is an étale Galois cover. We may then apply proposition 3.3.5.

□

3.3.3 Application to local cohomology

In the remainder of this section we apply the material from the two introductory paragraphs above to study the cohomology spaces $H^\bullet(\mathbb{A}^n_k \setminus Y)$. For this we define the following map.

Definition 3.3.7. Let $g \in k[x_1, \ldots, x_n]$ be a weighted homogeneous polynomial of degree d w.r.t. weights $(w_1, \ldots, w_n)$. Also assume that k that contains a unit root ζ_d of order d. We define a $\langle \zeta_d \rangle$-action on the algebra

$$S = \frac{k[x_1, \ldots, x_n, y_1, y_2]}{(g-1,\ y_1 y_2 - 1)}$$

as follows: the element ζ_d acts on S via the map

$$\begin{array}{rcl} x_i & \mapsto & \zeta_d^{w_i}\, x_i \\ y_1 & \mapsto & \zeta_d^{-1}\, y_1 \\ y_2 & \mapsto & \zeta_d\, y_2 \end{array}$$

Note that $\operatorname{Spec} S \cong Z_{\mathbb{A}^n_k}(g-1) \times (\mathbb{A}^1_k \setminus \{0\})$ and that the $\langle \zeta_d \rangle$-action described above is the product of the monodromy action with an action on $\mathbb{A}^1_k \setminus \{0\}$. We also define the algebra morphism

$$\begin{array}{rcl} \alpha\colon R & \to & S \\ x_i & \mapsto & x_i\, y_1^{w_i} \\ y & \mapsto & y_2^d \end{array}$$

where

$$R = \frac{k[x_1, \ldots, x_n, y]}{(gy-1)}.$$

By applying the Spec functor to α we obtain a morphism

$$\varphi\colon Z_{\mathbb{A}^n_k}(g-1) \times (\mathbb{A}^1_k \setminus \{0\}) \to \mathbb{A}^n_k \setminus Y$$

where as usual $Y = Z_{\mathbb{A}^n_k}(g)$. It is easy to verify that $\varphi \circ \phi_{\zeta_d} = \varphi$, where ϕ_{ζ_d} denotes the generator of the $\langle \zeta_d \rangle$-action on $Z_{\mathbb{A}^n_k}(g-1) \times (\mathbb{A}^1_k \setminus \{0\})$.

Note that the morphism φ can still be defined if $\zeta_d \notin k$. In this situation we only need to choose an extension $k' \supset k$ to define the $\langle \zeta_d \rangle$-action. But as we explained in paragraph 3.1.2, we can always consider the induced map on cohomology

$$H^\bullet(\varphi) \colon H^\bullet(\mathbb{A}^n_k \setminus Y) \to H^\bullet(Z_{\mathbb{A}^n_k}(g-1) \times (\mathbb{A}^1_k \setminus \{0\}))^{\langle \zeta_d \rangle}.$$

We start by proving a lemma about the algebra morphism α.

Proposition 3.3.8. *As an R-module, S is generated by the elements*

$$\{y_1^a y_2^b \in S \mid 0 \leq a, b \leq d-1\}.$$

Proof. Because of the relation $y_1 y_2 = 1$ it is enough to show that all the monomials of the form $\underline{x}^I y_j^m$ for $j \in \{1,2\}$ can be generated in this way. The element $x_i \in S$ can be written as $\alpha(x_i)\, y_2^{w_i}$. Since $y_2^d = \alpha(y)$ this takes care of the monomials of the form $\underline{x}^I y_2^m$. Now observe that in S,

$$y_1^d = y_1^d g(\underline{x}) = g(y_1^{w_1} x_1, \ldots, y_1^{w_n} x_n) = g(\alpha(x_1), \ldots, \alpha(x_n)) = \alpha(g(\underline{x})).$$

This takes care of the monomials of the form $\underline{x}^I y_1^m$. $\square$

In the proof of proposition 3.3.10 we will also need the following lemma.

Proposition 3.3.9. *Choose an integer $d \geq 2$ and let k be a field containing a unit root ζ_d of order d. This implies in particular that $char(k) \nmid d$. Let A be a k-algebra with a $\langle \zeta_d \rangle$-action that admits a presentation*

$$A = \frac{k[X_1, \ldots, X_n]}{I}$$

such that the generator ϕ_{ζ_d} can be written as $X_j \mapsto \zeta_d^{W_j} X_j$ with $1 \leq W_j \leq d-1$. Then $A^{\langle \zeta_d \rangle}$ is generated by the image in A of the set Ψ that consists of the monomials in the variables $X_1, \ldots, X_n$ that are weighted homogeneous of degree congruent to zero modulo d with respect to the weights $W_1, \ldots, W_n$.

Proof. It is obvious that $k[X_1, \ldots, X_n]$ also has a $\langle \zeta_d \rangle$-action and that every element of Ψ is invariant under this action. Now consider an element $e \in A^{\langle \zeta_d \rangle}$ that is represented by a polynomial

$$\sum_{J \in \mathbb{N}^n} a_J \, \underline{X}^J \in k[X_1, \ldots, X_n].$$

Then the polynomial

$$\frac{1}{d} \cdot \sum_{h \in \langle \zeta_d \rangle} \phi_h \left(\sum_{J \in \mathbb{N}^n} a_J \, \underline{X}^J \right) = \sum_{J \in \mathbb{N}^n} \frac{a_J}{d} \cdot \left(\sum_{h \in \langle \zeta_d \rangle} \phi_h(\underline{X}^J) \right)$$

represents e as well. If $\underline{X}^J \in \Psi$ then we have that $\frac{1}{d} \cdot \sum_{h \in \langle \zeta_d \rangle} \phi_h(\underline{X}^J) = \underline{X}^J$. On the other hand, if $\underline{X}^J \notin \Psi$ then we have $\phi_{\zeta_d}(\underline{X}^J) = \zeta_d^b \, \underline{X}^J$ for some $b \not\equiv 0 \bmod d$. It follows that

$$\sum_{h \in \langle \zeta_d \rangle} \phi_h(\underline{X}^J) = \underline{X}^J \cdot \sum_{i=0}^{d-1} (\zeta_d^b)^i = 0.$$

We see that $e \in A^{\langle \zeta_d \rangle}$ has a representation of the form

$$\sum_{\underline{X}^J \in \Psi} a_J \, \underline{X}^J$$

and this proves our claim. Note that in the end the coefficients a_J don't change; we only deleted the monomials that don't belong to Ψ. $\square$

Now we are ready to prove the main result for this section.

Proposition 3.3.10. *For every $i \geq 0$ the canonical map on rigid cohomology*

$$H^i(\varphi) \colon H^i(\mathbb{A}_k^n \setminus Y) \longrightarrow H^i(Z_{\mathbb{A}_k^n}(g-1) \times (\mathbb{A}_k^1 \setminus \{0\}))^{\langle \zeta_d \rangle}$$

is an isomorphism.

Proof. After base-changing to $\overline{k}$ the $\langle \zeta_d \rangle$-action on the closed points of the scheme $Z_{\mathbb{A}_{\overline{k}}^n}(g-1) \times (\mathbb{A}_{\overline{k}}^1 \setminus \{0\})$ can be written as:

$$\phi_{\zeta_d} \colon (a_1, \ldots, a_n, \lambda, \lambda^{-1}) \mapsto (\zeta_d^{w_1} \, a_1, \ldots, \zeta_d^{w_n} \, a_n, \zeta_d^{-1} \, \lambda, \zeta_d \, \lambda^{-1})$$

By looking at the last two coordinates it is immediately clear that the action of $\langle \zeta_d \rangle$ is free.

According to point ii) of proposition 3.3.6 it is therefore sufficient to verify that φ is the quotient map w.r.t. the action of $\langle \zeta_d \rangle$. This is equivalent to proving that α is isomorphic to the inclusion $S^{\langle \zeta_d \rangle} \hookrightarrow S$. That is, we need to show that α is injective and that its image is precisely $S^{\langle \zeta_d \rangle}$.

It is obvious that $\mathrm{Im}(\alpha) \subset S^{\langle \zeta_d \rangle}$, so we focus on the other inclusion. We know by proposition 3.3.8 that every element of $S^{\langle \zeta_d \rangle}$ has a presentation of the form

$$\sum_{0 \leq a,b \leq d-1} \widetilde{\alpha}(r_{a,b}) \, y_1^a y_2^b \in k[x_1, \ldots, x_n, y_1, y_2]$$

where the $r_{a,b}$ are elements of $k[x_1, \ldots, x_n, y]$ and $\widetilde{\alpha}$ is the morphism on polynomial rings that is defined through the same formulas as in definition 3.3.7.

Now consider the set Ψ as in the statement of proposition 3.3.9, with respect to the weights $(w_1, \ldots, w_n, d-1, 1)$. There may be pairs (a_0, b_0) for which $\widetilde{\alpha}(r_{a_0,b_0}) \neq 0$ but $y_1^{a_0} y_2^{b_0} \notin \Psi$. If this is the case then none of the monomials in $\widetilde{\alpha}(r_{a_0,b_0})\, y_1^{a_0} y_2^{b_0}$ lie in Ψ. According to proposition 3.3.9 we may then delete the entire term $\widetilde{\alpha}(r_{a_0,b_0})\, y_1^{a_0} y_2^{b_0}$ to obtain another representant. Therefore we may assume that for every pair (a, b) such that $\widetilde{\alpha}(r_{a,b}) \neq 0$ we have $y_1^a y_2^b \in \Psi$. But this precisely means that $a = b$. Because of the relation $y_1 y_2 = 1$ in S we may moreover take $a = b = 0$. The inclusion $S^{\langle \zeta_d \rangle} \subset \mathrm{Im}(\alpha)$ follows.

It remains to show that α is injective. If we write

$$R = k[x_1, \ldots, x_n, g^{-1}]$$

and

$$S = \frac{k[x_1, \ldots, x_n, y_1]}{(g-1)}[y_1^{-1}]$$

then α may be rewritten as

$$\frac{f}{g^t} \mapsto \frac{\overline{\beta}(f)}{(y_1^d)^t} \ , \ f \in k[x_1, \ldots, x_n], \ t \geq 0$$

where $\overline{\beta}$ is the composition of the map

$$\beta \colon k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n, y_1] \colon x_i \mapsto x_i\, y_1^{w_i}$$

with the quotient map

$$k[x_1, \ldots, x_n, y_1] \to \frac{k[x_1, \ldots, x_n, y_1]}{(g-1)}.$$

It is obvious that β is injective, so it suffices to show that

$$\mathrm{Im}(\beta) \cap (g-1) = (0).$$

To see this, assume that there exist nonzero polynomials $h \in k[x_1, \ldots, x_n, y_1]$ and $f \in k[T_1, \ldots, T_n]$ such that

$$(g(x_1, \ldots, x_n) - 1) \cdot h(x_1, \ldots, x_n, y_1) = f(x_1\, y_1^{w_1}, \ldots, x_n\, y_1^{w_n}). \qquad (3.3.1)$$

Now choose a term $m(\underline{x}, y_1)$ of h that has maximal degree. We may choose $m(\underline{x}, y_1)$ in such a way that there exists a non-constant term $m'(\underline{x})$ such that $m'(\underline{x})\, m(\underline{x}, y_1)$ appears as a term on the right-hand side of (3.3.1). But all the terms of $f(x_1\, y_1^{w_1}, \ldots, x_n\, y_1^{w_n})$ are of the form

$$c \cdot x_1^{a_1} \, \ldots \, x_n^{a_n}\, y_1^{a_1 w_1 + \ldots + a_n w_n}. \qquad (3.3.2)$$

In particular, $m(\underline{x}, y_1)$ is *not* of the form (3.3.2) and therefore it cancels out in the expression $g\,h - h$. Now write $g = \sum_i p_i(\underline{x})$ and $h = \sum_j q_j(\underline{x}, y_1)$ as sum of their terms. Also define

$$c_{ij} = \begin{cases} 1 & \text{if } p_i(\underline{x})\, q_j(\underline{x}, y_1) = c \cdot m(\underline{x}, y_1) \text{ for some constant } c \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

The fact that $m(\underline{x}, y_1)$ cancels out in the expression $g\,h - h$ implies that

$$\sum_{i,j} c_{ij} \cdot p_i(\underline{x})\, q_j(\underline{x}, y_1) = m(\underline{x}, y_1).$$

There must be at least one pair (i, j) such that $c_{ij} = 1$. For such a pair (i, j) we have that $m'(\underline{x})\, p_i(\underline{x})\, q_j(\underline{x}, y_1)$ is of the form (3.3.2). In this way we find another term $q_j(\underline{x}, y_1)$ of h that is *not* of the form (3.3.2). This term must also cancel out in $g\,h - h$. Also, $q_j(\underline{x}, y_1)$ is of strictly smaller degree than $m(\underline{x}, y_1)$ because g has no constant term. By continuing this process we find a term of h that is of minimal degree and that cancels out in $g\,h - h$. This is a contradiction, and we have shown that $\mathrm{Im}(\beta) \cap (g - 1) = (0)$. □

3.4 The cohomology of a certain ramified cover

In section 3.3 we have used the property that rigid cohomology behaves well with respect to étale Galois covers. In this section we will show that a similar property is true for a certain *ramified* cover. More specifically, consider the affine Milnor fiber $S \backslash S_\infty = Z_{\mathbb{A}^n_k}(g - 1)$ of a weighted homogeneous hypersurface singularity $Y = Z_{\mathbb{A}^n_k}(g)$. We will show in proposition 3.4.5 that the canonical map

$$H^\bullet_{rig}(S \setminus S_\infty)^{\langle \zeta_d \rangle} \longrightarrow \left(H^\bullet_{rig}(\widetilde{S} \setminus \widetilde{S}_\infty)^{\langle \zeta_d \rangle} \right)^{G(\underline{w})} \tag{3.4.1}$$

is an isomorphism. The difficulty in studying this map comes from the fact that the quotient map $\psi \colon \mathbb{P}^{n-1}_k \to \mathbb{P}_k(\underline{w}) = \mathbb{P}^{n-1}_k / G(\underline{w})$ is ramified. An easy way to see this is to observe that $G(\underline{w})$ does not act freely on $\mathbb{P}^{n-1}_k$. In fact, every point that lies on a coordinate hyperplane $x_i = 0$ corresponding to a weight $w_i > 1$ has a non-trivial stabilizer. It is then easy to see that the restriction $\widetilde{S} \setminus \widetilde{S}_\infty \to S \setminus S_\infty$ is still ramified. Because of this we cannot apply proposition 3.3.5 (except of course in the homogeneous case where $\underline{w} = (1, \dots, 1)$).

To prove that the map (3.4.1) is an isomorphism we are forced to use a more indirect approach. First we reduce our problem to a statement about algebraic de Rham cohomology with coefficients in K. The key idea here is to use the theorem of Baldassarri-Chiarellotto to circumvent all the problems that can in general arise when lifting to characteristic zero. After choosing an embedding $K \hookrightarrow \mathbb{C}$ we can then apply some results from the complex analytic theory of weighted projective hypersurfaces.

We start by giving an overview of the necessary complex theory in paragraph 3.4.1. As an easy application we prove the complex version of theorem 3.1.11 that we announced at the beginning of this chapter. We also argue that some of the classical results over $\mathbb{C}$ cannot be easily translated to rigid cohomology. The problem is that we don't know how a ramified cover generally behaves w.r.t. rigid cohomology. We will address this question in more detail in section 5.1.

In paragraph 3.4.2 we apply the complex theory to the study of the canonical map (3.4.1).

3.4.1 Results about Betti cohomology over $\mathbb{C}$

In this paragraph we discuss some classical results about the Betti cohomology of complex weighted homogeneous hypersurfaces. We argue that some of these results have no obvious analogues in rigid cohomology. However, these analytic properties will be needed in the next paragraph to prove a (weaker) result about rigid cohomology.

Throughout the remainder of this section we will use the notation $H^\bullet(X,\, \mathbb{C})$ for the Betti cohomology with complex coefficients of the complex topology of a $\mathbb{C}$-scheme X.

The proposition below is due to Steenbrink, Dolgachev and Dimca. It states that Betti cohomology is compatible with certain quotients by finite groups. This property can be proved more generally, see proposition 5.1.3 in chapter 5. However, the proof of proposition 3.4.1 below has the advantage that it is "as algebraic as possible". This then makes it clear which parts of the proof can be translated to rigid cohomology, and for which parts this is not possible. Also see remark 3.4.2 below.

Proposition 3.4.1. *Consider a polynomial $g \in \mathbb{C}[x_1, \dots, x_n]$ that is weighted homogeneous of degree d w.r.t. weights $\underline{w}$. Define as before $S = Z_{\mathbb{P}_\mathbb{C}(1,\underline{w})}(g - x_0^d)$ and $S_\infty = Z_{\mathbb{P}_\mathbb{C}(\underline{w})}(g)$ together with the usual action of the group $\langle \zeta_d \rangle$. Also assume that S_∞ (hence also S) is quasi-smooth. Then there is an isomorphism*

$$H^{n-1}(\mathbb{P}_\mathbb{C}(\underline{w}) \setminus S_\infty,\, \mathbb{C}) \xrightarrow{\sim} H^{n-1}(S \setminus S_\infty,\, \mathbb{C})^{\langle \zeta_d \rangle}.$$

If moreover $\widetilde{S}_\infty$ (hence also $\widetilde{S}$) is smooth then there is an isomorphism

$$H^{n-1}(\mathbb{P}_\mathbb{C}(\underline{w}) \setminus S_\infty,\, \mathbb{C}) \xrightarrow{\sim} H^{n-1}(\mathbb{P}^{n-1}_\mathbb{C} \setminus \widetilde{S}_\infty,\, \mathbb{C})^{G(\underline{w})}.$$

Proof. It is a classical result by Steenbrink [Ste77a] that the Betti cohomology spaces $H^i(S,\, \mathbb{C})$ of a projective V-manifold (in particular, of a quasi-smooth weighted projective hypersurface) S admit a Hodge-type decomposition

$$H^i(S,\, \mathbb{C}) \xrightarrow{\sim} \bigoplus_{p+q=i} H^q(S,\, \widetilde{\Omega}^p_S). \tag{3.4.2}$$

In this decomposition the sheaves $\widetilde{\Omega}_S^\bullet$ are defined as $\widetilde{\Omega}_S^p := j_*\Omega_W^p$, where $j\colon W \to S$ is the inclusion of the smooth locus of S. These sheaves are called the *modified differential sheaves* on S. Also see paragraph 4.1 in [Dol82] for a different characterization of these sheaves. A similar decomposition as (3.4.2) also holds for the Betti cohomology of S_∞.

Implicitly we are also using Serre's GAGA theorem [Ser56, Théorème 1]. The sheaves appearing in [Ste77a, Theorem 1.12] live on the analytification of S, but the modified differentials in the decomposition (3.4.2) are *algebraic.*

Another crucial ingredient is the Poincaré duality theorem for rational homology manifolds (which include V-manifolds). See for instance [Bor57]. This theorem can be used to obtain a Gysin-type sequence that is dual to the sequence on cohomology with compact supports:

$$\ldots \to H^i(S \setminus S_\infty, \mathbb{C}) \to H^{i-1}(S_\infty, \mathbb{C}) \to H^{i+1}(S, \mathbb{C}) \to \ldots \tag{3.4.3}$$

The cohomology of the sheaves $\widetilde{\Omega}_S^p$ for a quasi-smooth hypersurface S is studied in detail in section 4 of [Dol82]. In paragraph 4.4 it is shown that one can use the resolution from [Dol82, Proposition 4.1.7] together with the decomposition (3.4.2) and the long exact sequence (3.4.3) to obtain an isomorphism

$$H^{n-1}(S \setminus S_\infty, \mathbb{C}) \xrightarrow{\sim} H^{n-2}(S_\infty, \mathbb{C})_{prim} \oplus H^{n-1}(S, \mathbb{C})_{prim}. \tag{3.4.4}$$

The subscripts *prim* indicate the primitive parts of the cohomology; see [Dol82] for the precise definitions.

It has been shown in [Dim90b, Remark 1.21] that the $\langle\zeta_d\rangle$-invariant subspace of $H^{n-1}(S, \mathbb{C})_{prim}$ is zero. So by taking the monodromy invariants of (3.4.4) we obtain an isomorphism

$$H^{n-1}(S \setminus S_\infty, \mathbb{C})^{\langle\zeta_d\rangle} \xrightarrow{\sim} H^{n-2}(S_\infty, \mathbb{C})_{prim}.$$

On the other hand we have an isomorphism

$$H^{n-2}(S_\infty, \mathbb{C})_{prim} \xrightarrow{\sim} H^{n-1}(\mathbb{P}_{\mathbb{C}}(\underline{w}) \setminus S_\infty, \mathbb{C}).$$

This is an easy consequence of the Gysin sequence for $S_\infty \subset \mathbb{P}_{\mathbb{C}}(\underline{w})$ and the weak Lefschetz theorem for weighted projective hypersurfaces, which is described in [Dol82, Theorem 4.2.2]. By combining the two isomorphisms above we obtain the first claim of the proposition.

It remains to show the second statement. By [Dol82, Proposition 2.2.3] there is an isomorphism

$$\widetilde{\Omega}^\bullet_{\mathbb{P}_{\mathbb{C}}(\underline{w})} \xrightarrow{\sim} \psi_*^{G(\underline{w})}\left(\Omega^\bullet_{\mathbb{P}^{n-1}_{\mathbb{C}}}\right)$$

where $\widetilde{\Omega}^\bullet_{\mathbb{P}_{\mathbb{C}}(\underline{w})}$ denote the modified differential sheaves on $\mathbb{P}_{\mathbb{C}}(\underline{w})$ and $\psi\colon \mathbb{P}^{n-1}_{\mathbb{C}} \to \mathbb{P}_{\mathbb{C}}(\underline{w})$ is the quotient map. Combining this with the decomposition (3.4.2) and the resolution given in [Dol82, Proposition 4.1.7] we obtain isomorphisms

$$H^\bullet(\mathbb{P}_{\mathbb{C}}(\underline{w}),\, \mathbb{C}) \xrightarrow{\sim} H^\bullet(\mathbb{P}^{n-1}_{\mathbb{C}},\, \mathbb{C})^{G(\underline{w})}.$$

and

$$H^\bullet(S_\infty,\, \mathbb{C}) \xrightarrow{\sim} H^\bullet(\widetilde{S}_\infty,\, \mathbb{C})^{G(\underline{w})}.$$

The second claim of the proposition now follows by using the Gysin sequences for $S_\infty \subset \mathbb{P}_{\mathbb{C}}(\underline{w})$ and $\widetilde{S}_\infty \subset \mathbb{P}^{n-1}_{\mathbb{C}}$ together with the *five lemma*. □

It can be verified that the isomorphisms in the proposition above are compatible with the canonical mixed Hodge structure, at least up to a twist.

Remark 3.4.2. Certain parts of the proof of proposition 3.4.1 do not have a direct counterpart in rigid cohomology.

Of course it is possible to define the sheaves $\widetilde{\Omega}^\bullet_{\mathcal{X}}$ for a quasi-smooth weighted projective hypersurface $\mathcal{X}$ over any base field K. The Hodge cohomology can be defined formally as

$$H^i_{\mathrm{formal}}(\mathcal{X}) = \bigoplus_{p+q=i} H^q(\mathcal{X},\, \widetilde{\Omega}^p_{\mathcal{X}}).$$

It is then possible to express the primitive part of $H^i_{\mathrm{formal}}(\mathcal{X})$ in terms of *formal* differentials. At least, this is the case if K is of characteristic zero. For example, if we write $\mathcal{X} = Z(\mathcal{G}) \subset \mathbb{P}_K(w_1, \ldots, w_n)$, then the formal $(n-1)$-forms are of the shape

$$\frac{A\,\Omega}{\mathcal{G}^t},$$

where A is a weighted homogeneous polynomial of degree $t \cdot \deg_{\underline{w}} \mathcal{G} - \sum_{i=1}^n w_i$. This is worked out in detail in paragraph 4.2 of [Dol82]. Also see [Dim92, Proposition 6.1.21].

However, assume that $K = \operatorname{Frac} \mathcal{V}$ is an ultrametric field of mixed characteristic, with residue $k = \mathcal{V}/(\pi)$, and write $X = \mathcal{X}_k$. Then there is no obvious connection between $H^\bullet_{rig}(X)$ and $H^\bullet_{\mathrm{formal}}(\mathcal{X})$. Indeed, the Hodge-type decomposition (3.4.2) heavily relies on a modification of the holomorphic Poincaré lemma, see [Ste77a, Corollary 1.9]. This argument completely breaks down for rigid cohomology. Another difficulty is that the Poincaré duality theorem for V-manifolds is only known for Betti cohomology. The proof in [Bor57] really uses that the spaces involved are Hausdorff. For these reasons it seems difficult to prove an analogue of proposition 3.4.1 for rigid cohomology.

On the other hand, we will prove an analogous statement in the case where $\underline{w} = (1, \ldots, 1)$. See proposition 3.5.7 in the next section. We can do this because in the homogeneous case all the schemes involved are smooth and affine. In this situation the theory of Monsky-Washnitzer and Baldassarri-Chiarellotto gives a relation between rigid cohomology and *classical* differential forms. Also, we can use the Poincaré duality theorem for the rigid cohomology of *smooth* schemes (in the form of the Gysin sequence (1.2.20)). A similar remark applies to proposition 3.4.5 below.

As a corollary of proposition 3.4.1 we prove the complex counterpart to our theorem 3.1.11.

Proposition 3.4.3. *Use the notations from proposition 3.4.1. Also assume that g defines an isolated hypersurface singularity $Y = Z_{\mathbb{A}^n_{\mathbb{C}}}(g)$. Then there is an isomorphism (of mixed Hodge structures)*

$$H^n_{\{0\}}(Y, \mathbb{C}) \xrightarrow{\sim} H^{n-1}(\mathbb{P}_{\mathbb{C}}(\underline{w}) \setminus S_\infty, \mathbb{C}).$$

If moreover $\widetilde{S}_\infty$ is smooth then we have an isomorphism

$$H^n_{\{0\}}(Y, \mathbb{C}) \xrightarrow{\sim} H^{n-1}(\mathbb{P}^{n-1}_{\mathbb{C}} \setminus \widetilde{S}_\infty, \mathbb{C})^{G(\underline{w})}.$$

Proof. The assumption that Y has an isolated singularity at the origin implies that S_∞ (hence also S) is quasi-smooth. See proposition 3.1.6. By looking at the Gysin sequence for $S_\infty \subset S$ one can show that $H^i(S \setminus S_\infty, \mathbb{C}) = 0$ for $i \notin \{0, n-1\}$.

Our results from sections 3.2 and 3.3 also hold for Betti cohomology, in fact the proofs are much simpler. After combining proposition 3.3.10 with the Künneth formula and the observation above we obtain an isomorphism

$$H^n_{\{0\}}(Y, \mathbb{C}) \xrightarrow{\sim} H^{n-1}(S \setminus S_\infty, \mathbb{C})^{\langle \zeta_d \rangle}.$$

The result now follows from proposition 3.4.1. □

In paragraph 3.1.1 we have explained that for the theory over $\mathbb{C}$ one expects the right condition for theorem 3.1.11 to be the quasi-smoothness of S_∞. We have now shown that this is indeed the case. In definition 3.1.7 we have imposed the stronger condition that $\widetilde{S}_\infty$ is smooth. The reason for this assumption should now be clear. The situation can be summed up as follows: we do not want to pass to the quotient $(S \setminus S_\infty)/\langle \zeta_d \rangle \cong \mathbb{P}_k(\underline{w}) \setminus S_\infty$. The reason is that we cannot say much about the rigid cohomology of weighted projective spaces.

As another application of proposition 3.4.1 we prove a complex version of the result that we announced in the introduction of this section.

Proposition 3.4.4. *Use the same notation as in proposition 3.4.1. Also assume that $\widetilde{S}_\infty$ is smooth. Then we have an isomorphism*

$$H^{n-1}(S \setminus S_\infty, \mathbb{C})^{\langle \zeta_d \rangle} \xrightarrow{\sim} \left(H^{n-1}(\widetilde{S} \setminus \widetilde{S}_\infty, \mathbb{C})^{\langle \zeta_d \rangle} \right)^{G(\underline{w})}.$$

Proof. This follows immediately by applying proposition 3.4.1 to both g and $\widetilde{g}$. $\square$

3.4.2 Application to rigid cohomology

In this paragraph we prove that the canonical map (3.4.1) is an isomorphism. We start by giving a careful definition of this map.

As usual, let g denote a weighted homogeneous polynomial of degree d and weights $\underline{w}$. The quotient map $\psi\colon \widetilde{S}\setminus\widetilde{S}_\infty \to S\setminus S_\infty$ is the same as the Spec of the composition of algebra morphisms

$$\tfrac{k[x_1,\dots,x_n]}{(g-1)} \xrightarrow[\cong]{\tau} \left(\tfrac{k[x_1,\dots,x_n]}{(\widetilde{g}-1)}\right)^{G(\underline{w})} \hookrightarrow \tfrac{k[x_1,\dots,x_n]}{(\widetilde{g}-1)}$$

where the morphism τ is defined by $\tau(x_i) = x_i^{w_i}$. Now let ϕ_{ζ_d} (resp. $\widetilde{\phi}_{\zeta_d}$) denote the algebra morphism given by $x_i \mapsto \zeta_d^{w_i}x_i$ (resp. by $x_i \mapsto \zeta_d x_i$). Then we clearly have an identity $\tau\circ\phi_{\zeta_d} = \widetilde{\phi}_{\zeta_d}\circ\tau$. From this we see that the quotient map ψ is both $\langle\zeta_d\rangle$-equivariant and $G(\underline{w})$-equivariant (here we have applied definition 3.3.4 to both g and $\widetilde{g}$). As a result, the map $H^\bullet_{rig}(\psi)$ on cohomology will send a $\langle\zeta_d\rangle$-invariant element of $H^\bullet_{rig}(S\setminus S_\infty)$ to an element of $H^\bullet_{rig}(\widetilde{S}\setminus\widetilde{S}_\infty)$ that is invariant under both $\langle\zeta_d\rangle$ and $G(\underline{w})$. Since the actions of these two groups commute with each other we can also write this restriction as

$$H^\bullet_{rig}(S\setminus S_\infty)^{\langle\zeta_d\rangle} \longrightarrow \left(H^\bullet_{rig}(\widetilde{S}\setminus\widetilde{S}_\infty)^{\langle\zeta_d\rangle}\right)^{G(\underline{w})}. \tag{3.4.5}$$

That is: the group $G(\underline{w})$ acts on $H^\bullet_{rig}(\widetilde{S}\setminus\widetilde{S}_\infty)^{\langle\zeta_d\rangle}$ and the $G(\underline{w})$-invariants are precisely the elements of $H^\bullet_{rig}(\widetilde{S}\setminus\widetilde{S}_\infty)$ that are invariant under both $G(\underline{w})$ and $\langle\zeta_d\rangle$.

In the proof of proposition 3.4.5 below we will also need a good lift of the hypersurface $\widetilde{S}_\infty = Z(\widetilde{g}) \subset \mathbb{P}^{n-1}_k$. Let us assume that this hypersurface is smooth. Then it is easy to see that $\widetilde{S}_\infty$ can be lifted to a smooth hypersurface $\widetilde{\mathcal{S}}_\infty = Z(\widetilde{\mathcal{G}}) \subset \mathbb{P}^{n-1}_{\mathcal{V}}$. More specifically, we can do this by only lifting the nonzero coefficients of $\widetilde{g}$. As the notation suggests, the polynomial $\widetilde{\mathcal{G}}$ is then indeed the same thing as applying the tilde operator to a weighted homogeneous polynomial $\mathcal{G}$ that lifts g. The weighted projective hypersurface $\mathcal{S}_\infty = Z(\mathcal{G}) \subset \mathbb{P}^{n-1}_{\mathcal{V}}$ is a quasi-smooth lift of S_∞. Similarly we define the hypersurfaces $\widetilde{\mathcal{S}}$ and $\mathcal{S}$ by the equations $\widetilde{\mathcal{G}} - x_0^d$ and $\mathcal{G} - x_0^d$ respectively.

It is clear that the actions of the groups $\langle\zeta_d\rangle$ and $G(\underline{w})$ also lift. Indeed, if $\zeta\in k$ is a unit root of order r then its Teichmüller lift is a unit root of the same order in $\mathcal{V}$. The relevant group actions can then be defined analogously as in paragraphs 3.1.2 and 3.3.1. In this way we also obtain a quotient map $\Psi\colon \widetilde{\mathcal{S}}\setminus\widetilde{\mathcal{S}}_\infty \to \mathcal{S}\setminus\mathcal{S}_\infty$ that lifts the quotient map ψ.

Proposition 3.4.5. *Let $g \in k[x_1,\dots,x_n]$ be a weighted homogeneous polynomial of degree d with respect to weights $\underline{w}$. We assume that g satisfies all* *the assumptions of definition 3.1.7, in particular that $\widetilde{S}_\infty$ is smooth. Then the canonical map*

$$H^{n-1}_{rig}(\psi)\colon H^{n-1}_{rig}(S \setminus S_\infty)^{\langle\zeta_d\rangle} \longrightarrow \left(H^{n-1}_{rig}(\widetilde{S} \setminus \widetilde{S}_\infty)^{\langle\zeta_d\rangle}\right)^{G(\underline{w})} \tag{3.4.6}$$

is an isomorphism.

Proof. Since $S \setminus S_\infty$ and $\widetilde{S} \setminus \widetilde{S}_\infty$ are both smooth and affine we know by proposition 3.3.6 that the map is injective. It remains to show that (3.4.6) is surjective.

By our assumptions $\widetilde{S}$ and $\widetilde{S}_\infty$ are smooth projective hypersurfaces and by theorem 1.2.2 (Baldassarri-Chiarellotto) we have an isomorphism of K-vector spaces

$$H^\bullet_{dR}(\widetilde{\mathcal{S}}_K \setminus (\widetilde{\mathcal{S}}_K)_\infty) \xrightarrow{\sim} H^\bullet_{rig}(\widetilde{S} \setminus \widetilde{S}_\infty).$$

The subscript K on the left-hand side denotes the generic fiber of the lifts that we have constructed above. Since the actions of $G(\underline{w})$ and $\langle\zeta_d\rangle$ lift to $\mathcal{V}$ and since all the maps involved are canonical we now find a commutative diagram as follows:

$$\begin{tikzcd}
H^\bullet_{dR}(\mathcal{S}_K \setminus (\mathcal{S}_K)_\infty)^{\langle\zeta_d\rangle} \arrow[r, "H^\bullet_{dR}(\Psi_K)"] \arrow[d] & \left(H^\bullet_{dR}(\widetilde{\mathcal{S}}_K \setminus (\widetilde{\mathcal{S}}_K)_\infty)^{\langle\zeta_d\rangle}\right)^{G(\underline{w})} \arrow[d, "\cong"] \\
H^\bullet_{rig}(S \setminus S_\infty)^{\langle\zeta_d\rangle} \arrow[r, "H^\bullet_{rig}(\psi)"] & \left(H^\bullet_{rig}(\widetilde{S} \setminus \widetilde{S}_\infty)^{\langle\zeta_d\rangle}\right)^{G(\underline{w})}
\end{tikzcd}$$

In order to show that $H^{n-1}_{rig}(\psi)$ is surjective it suffices to show that $H^{n-1}_{dR}(\Psi_K)$ is an isomorphism.

The injectivity of $H^{n-1}_{dR}(\Psi_K)$ follows from the existence of a trace map on differentials. This is similar to (and easier than) the proof of [Ber97b, Proposition 3.6]. Also see the construction in the proof of [vdP86, Proposition 3.1].

It is now sufficient to show that

$$\dim_K H^{n-1}_{dR}(\mathcal{S}_K \setminus (\mathcal{S}_K)_\infty)^{\langle\zeta_d\rangle} = \dim_K \left(H^{n-1}_{dR}(\widetilde{\mathcal{S}}_K \setminus (\widetilde{\mathcal{S}}_K)_\infty)^{\langle\zeta_d\rangle}\right)^{G(\underline{w})} \tag{3.4.7}$$

For this we use the assumption (from the beginning of this chapter) that our base field k is finite. Then we can assume that K is an algebraic extension of a p-adic field $\mathbb{Q}_p$, hence $K \subset \mathbb{C}_p$. After choosing an isomorphism $\mathbb{C}_p \cong \mathbb{C}$ this gives us an embedding $K \hookrightarrow \mathbb{C}$. We may then base-change to $\mathbb{C}$ and apply Grothendieck's theorem [Gro66, Theorem 1] to obtain equalities

$$\begin{aligned}
\dim_K H^{n-1}_{dR}(\mathcal{S}_K \setminus (\mathcal{S}_K)_\infty)^{\langle\zeta_d\rangle} &= \dim_{\mathbb{C}} H^{n-1}_{dR}(\mathcal{S}_{\mathbb{C}} \setminus (\mathcal{S}_{\mathbb{C}})_\infty)^{\langle\zeta_d\rangle} \\
&= \dim_{\mathbb{C}} H^{n-1}(\mathcal{S}_{\mathbb{C}} \setminus (\mathcal{S}_{\mathbb{C}})_\infty, \mathbb{C})^{\langle\zeta_d\rangle}
\end{aligned}$$

and

$$\begin{aligned}\dim_K \left(H^{n-1}_{dR}(\widetilde{\mathcal{S}}_K \setminus (\widetilde{\mathcal{S}}_K)_\infty)^{\langle\zeta_d\rangle}\right)^{G(\underline{w})} &= \dim_{\mathbb{C}} \left(H^{n-1}_{dR}(\widetilde{\mathcal{S}}_{\mathbb{C}} \setminus (\widetilde{\mathcal{S}}_{\mathbb{C}})_\infty)^{\langle\zeta_d\rangle}\right)^{G(\underline{w})} \\ &= \dim_{\mathbb{C}} \left(H^{n-1}(\widetilde{\mathcal{S}}_{\mathbb{C}} \setminus (\widetilde{\mathcal{S}}_{\mathbb{C}})_\infty, \mathbb{C})^{\langle\zeta_d\rangle}\right)^{G(\underline{w})}.\end{aligned}$$

The equality (3.4.7) then follows from the isomorphism

$$H^{n-1}(\mathcal{S}_{\mathbb{C}} \setminus (\mathcal{S}_{\mathbb{C}})_\infty, \mathbb{C})^{\langle\zeta_d\rangle} \xrightarrow{\sim} \left(H^{n-1}(\widetilde{\mathcal{S}}_{\mathbb{C}} \setminus (\widetilde{\mathcal{S}}_{\mathbb{C}})_\infty, \mathbb{C})^{\langle\zeta_d\rangle}\right)^{G(\underline{w})}$$

that we proved in proposition 3.4.4. □

The only reason why we assumed our ground field k to be finite is that this guarantees the existence of an embedding $K \hookrightarrow \mathbb{C}$, which allows us to use complex methods. We remark that the proof could also have been carried out without complex methods. Indeed, one can use the following property:

Proposition 3.4.6. *Let X be a smooth affine K-scheme with an action of a finite group G. Consider the quotient map $\psi\colon X \to Y = X/G$ and assume that Y is again smooth. Then the canonical map $\Omega^\bullet_Y \to \psi^G_*(\Omega^\bullet_X)$ is an isomorphism.*

Proof. Since X and Y are smooth their associated differential sheaves are locally free. On affine open subsets where these sheaves are free the argument is similar to [Dol82, Proposition 2.2.2]. The proposition then follows from a (tedious) glueing argument. □

This proposition can be used to obtain a more direct proof of the equality (3.4.7). Indeed, by taking global sections one finds an isomorphism

$$\Gamma(Y, \Omega^\bullet_Y) \xrightarrow{\sim} \Gamma(Y, \psi^G_*(\Omega^\bullet_X)) \cong \Gamma(X, \Omega^\bullet_X)^G,$$

which then gives an isomorphism on the algebraic de Rham cohomology.

We preferred to use the complex method because this allowed us to prove the expected complex version of theorem 3.1.11. Also, the remainder of the proof of theorem 3.1.11 will use a rigid cohomology analogue of proposition 3.4.1 in the *homogeneous* case. The proof of this property (see proposition 3.5.7) uses formally the same ideas as the complex property.

3.5 Proof of the theorem

This section contains the proof of theorem 3.1.11.

In paragraph 3.5.1 we start by proving a few results about the rigid cohomology of smooth hypersurfaces. The statements in this first paragraph are all rather standard, but we are not aware of a good reference.

We proceed by showing that the cohomology space $H^n(\mathbb{P}^n_k \setminus \widetilde{S})$ has no monodromy-invariants. Combining this with the short exact sequence (3.5.5) from the first paragraph we obtain an analogue of proposition 3.4.1 for the rigid cohomology of smooth projective hypersurfaces.

In paragraph 3.5.3 we prove theorem 3.1.11 by combining the results of all the previous sections.

Throughout this section we use the notations from theorem 3.1.11, although paragraph 3.5.1 is about general smooth hypersurfaces.

3.5.1 Cohomology of smooth projective hypersurfaces

We start by recalling a few facts about the rigid cohomology of smooth hypersurfaces.

First we consider the rigid cohomology of a weighted projective space $\mathbb{P}^n_k$. It is well-known that $H^i(\mathbb{P}^n_k)$ is zero for i odd. For i even $H^i(\mathbb{P}^n_k)$ is one-dimensional with Frobenius acting as $q^{\frac{i}{2}}\sigma$. This can easily be shown with the Gysin sequence (1.2.20), using induction on n.

It is moreover known that the rigid cohomology $H^i(\mathbb{P}^n_k)$ for i even is generated by the *cycle class* of a linear subspace $V \subset \mathbb{P}^n_k$ of codimension $i/2$. Also, for a smooth hypersurface $\widetilde{S} \subset \mathbb{P}^n_k$ and $i \leq 2n-2$, the cycle class of the intersection $V \cap \widetilde{S}$ defines a nonzero class of $H^i(\widetilde{S})$. The notion of cycle classes in rigid cohomology was introduced in [Pet03].

This information about $H^i(\mathbb{P}^n_k)$ can be used to prove the following statements about the rigid cohomology of a smooth projective hypersurface.

Proposition 3.5.1. *Let $\widetilde{S} \subset \mathbb{P}^n_k$ with $n \geq 2$ be a smooth projective hypersurface. Then the following properties hold:*

i) The Gysin map $H^{i-2}(\widetilde{S})(-1) \to H^i(\mathbb{P}^n_k)$ is an isomorphism for $i \notin \{0, n+1\}$.

ii) For $i \notin \{0, n\}$ we have $H^i(\mathbb{P}^n_k \setminus \widetilde{S}) = 0$.

iii) There is a short exact sequence

$$0 \longrightarrow H^n(\mathbb{P}^n_k \setminus \widetilde{S}) \longrightarrow H^{n-1}(\widetilde{S})(-1) \longrightarrow H^{n+1}(\mathbb{P}^n_k) \longrightarrow 0 \qquad (3.5.1)$$

If n is even then this sequence gives us an isomorphism $H^n(\mathbb{P}^n_k \setminus \widetilde{S}) \xrightarrow{\sim} H^{n-1}(\widetilde{S})(-1)$.

Proof. The short exact sequence (3.5.1) is simply what remains of the Gysin sequence (1.2.20) after applying points i) and ii). If n is even then $H^{n+1}(\mathbb{P}^n_k) = 0$ and the short exact sequence becomes an isomorphism.

Since $\mathbb{P}^n_k \setminus \widetilde{S}$ is smooth affine of dimension n we know that $H^i(\mathbb{P}^n_k \setminus \widetilde{S}) = 0$ for $i > n$. This implies that the maps $H^{i-2}(\widetilde{S})(-1) \to H^i(\mathbb{P}^n_k)$ are isomorphisms for $i > n+1$.

It remains to prove point i) for $0 < i < n+1$. For any i in this range, the Gysin map $H^{i-2}(\widetilde{S})(-1) \to H^i(\mathbb{P}^n_k)$ is dual to the map $H^{2n-i}_c(\mathbb{P}^n_k) \to H^{2n-i}_c(\widetilde{S})$ on rigid cohomology with compact supports. Since $\mathbb{P}^n_k$ and $\widetilde{S}$ are proper we may drop the compact supports. Therefore it is sufficient to show that the canonical map $H^i(\mathbb{P}^n_k) \to H^i(\widetilde{S})$ is an isomorphism for $n-1 < i < 2n$. By our earlier computations we know that in this range we have an equality of dimensions

$$\dim H^i(\widetilde{S}) = \dim H^{i+2}(\mathbb{P}^n_k) = \begin{cases} 1 & \text{for } i \text{ even} \\ 0 & \text{for } i \text{ odd} \end{cases}$$

So it remains to show that the map $H^i(\mathbb{P}^n_k) \to H^i(\widetilde{S})$ is nonzero when $i < 2n$ and i is even. For this we may of course assume that k is algebraically closed. Then we can choose a linear subspace $V \subset \mathbb{P}^n_k$ of codimension $i/2$ such that $V \cap \widetilde{S}$ is smooth. As noted before, V (resp. $V \cap \widetilde{S}$) defines a cycle class $c \in H^i(\mathbb{P}^n_k)$ (resp. $c' \in H^i(\widetilde{S})$). It can be deduced from [Pet03, Proposition 7.5] that c and c' are nonzero. According to [Pet03, Proposition 7.7] we have that $c \mapsto c'$.

This concludes the proof of point i). Point ii) simply follows from the Gysin sequence. $\square$

We proceed with a lemma about the rigid cohomology of the hypersurface at infinity of $\widetilde{S}$.

Proposition 3.5.2. *Use the same notations as in proposition 3.5.1. Let $\widetilde{S}_\infty \subset \widetilde{S}$ denote the intersection of $\widetilde{S}$ with the hyperplane at infinity. If $\widetilde{S}_\infty$ is smooth then the following properties hold:*

i) The Gysin map $\alpha_i\colon H^{i-2}(\widetilde{S}_\infty)(-1) \to H^i(\widetilde{S})$ is an isomorphism for $i \notin \{0, n-1, n\}$.

ii) For $i \notin \{0, n-1\}$ we have $H^i(\widetilde{S} \setminus \widetilde{S}_\infty) = 0$.

iii) There is a short exact sequence

$$0 \longrightarrow \mathit{Coker}(\alpha_{n-1}) \longrightarrow H^{n-1}(\widetilde{S} \setminus \widetilde{S}_\infty) \longrightarrow \mathit{Ker}(\alpha_n) \longrightarrow 0 \qquad (3.5.2)$$

Proof. We have a diagram of closed immersions as follows:

$$\begin{array}{ccc} \widetilde{S}_\infty & \longrightarrow & \mathbb{P}^{n-1}_k \\ \downarrow & & \downarrow \\ \widetilde{S} & \longrightarrow & \mathbb{P}^n_k \end{array} \qquad (3.5.3)$$

Since rigid cohomology with compact supports is functorial w.r.t. proper morphisms, we may apply $H^\bullet_c$ to the diagram above. After using Poincaré duality we obtain a commutative diagram

$$\begin{tikzcd}
H^{i-4}(\widetilde{S}_\infty)(-2) \arrow[r, "\gamma_{i-2}"] \arrow[d, "\alpha_{i-2}"'] & H^{i-2}(\mathbb{P}^{n-1}_k)(-1) \arrow[d, "\beta_i"] \\
H^{i-2}(\widetilde{S})(-1) \arrow[r, "\delta_i"] & H^i(\mathbb{P}^n_k)
\end{tikzcd} \tag{3.5.4}$$

where the arrows are precisely the Gysin maps associated to the immersions from (3.5.3).

It follows from proposition 3.5.1 that δ_i is an isomorphism for $i \notin \{0, n+1\}$. In a similar way the map $\gamma_i \colon H^{i-2}(\widetilde{S}_\infty) \to H^i(\mathbb{P}^{n-1}_k)$ is an isomorphism for $i \notin \{0, n\}$. It is also clear that the maps β_i are isomorphisms: this follows directly from the Gysin sequence for $\mathbb{P}^{n-1}_k \subset \mathbb{P}^n_k$.

As a consequence, we find that α_i is an isomorphism for $i \notin \{0, n-1, n\}$, proving point i). We also see that α_{n-1} is injective.

By considering the Gysin sequence for $\widetilde{S}_\infty \subset \widetilde{S}$ we see that $H^i(\widetilde{S} \setminus \widetilde{S}_\infty) = 0$ for $i \notin \{0, n-1, n\}$. But $\widetilde{S} \setminus \widetilde{S}_\infty$ is a smooth affine scheme of dimension $n-1$. It follows that $H^i(\widetilde{S} \setminus \widetilde{S}_\infty) = 0$ for $i > n-1$. This proves point ii). By plugging this back into the Gysin sequence we also find that α_n is surjective.

The short exact sequence from iii) follows directly from the Gysin sequence for $\widetilde{S}_\infty \subset \widetilde{S}$, using the fact that α_{n-1} is injective and that α_n is surjective. $\square$

The short exact sequence (3.5.2) can be refined as follows.

Proposition 3.5.3. *Use the same notations as in proposition 3.5.2. Then we have a Frobenius-equivariant short exact sequence*

$$0 \longrightarrow H^n(\mathbb{P}^n_k \setminus \widetilde{S})(+1) \longrightarrow H^{n-1}(\widetilde{S} \setminus \widetilde{S}_\infty) \longrightarrow H^{n-1}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty) \longrightarrow 0 \tag{3.5.5}$$

Proof. We have to show that $\mathrm{Coker}(\alpha_{n-1}) \cong H^n(\mathbb{P}^n_k \setminus \widetilde{S})(+1)$ and $\mathrm{Ker}(\alpha_n) \cong H^{n-1}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$.

Case I: n is even In this case $H^{n-1}(\mathbb{P}^{n-1}_k) = 0$. Since γ_{n-1} is an isomorphism we read from diagram (3.5.4) that $\mathrm{Coker}(\alpha_{n-1}) \cong H^{n-1}(\widetilde{S})$. Combining this with point iii) of proposition 3.5.1 we obtain an isomorphism

$$\mathrm{Coker}(\alpha_{n-1}) \xrightarrow{\sim} H^n(\mathbb{P}^n_k \setminus \widetilde{S})(+1).$$

Since δ_{n+2} is an isomorphism it follows from the diagram (3.5.4) that $\mathrm{Ker}(\alpha_n) \cong \mathrm{Ker}(\gamma_n)$. From the short exact sequence

$$0 \longrightarrow H^{n-1}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty) \longrightarrow H^{n-2}(\widetilde{S}_\infty)(-1) \xrightarrow{\gamma_n} H^n(\mathbb{P}^{n-1}_k) \longrightarrow 0$$

we obtain an isomorphism

$$\mathrm{Ker}(\gamma_n) \cong H^{n-1}(\mathbb{P}_k^{n-1} \setminus \widetilde{S}_\infty).$$

Case II: n **is odd** In this case $H^{n-2}(\mathbb{P}_k^{n-1} \setminus \widetilde{S}_\infty) = 0$ and $H^n(\mathbb{P}_k^{n-1}) = 0$. By looking at the Gysin sequence for $\widetilde{S}_\infty \subset \mathbb{P}_k^{n-1}$ and combining this with the isomorphism $H^{n-1}(\mathbb{P}_k^{n-1} \setminus \widetilde{S}_\infty) \xrightarrow{\sim} H^{n-2}(\widetilde{S}_\infty)(-1)$ from proposition 3.5.1 we see that γ_{n-1} is an isomorphism. Also, we know that δ_{n+1} is surjective. From the diagram (3.5.4) we then deduce an isomorphism

$$\overline{\alpha_{n-1}}\colon H^{n-3}(\widetilde{S}_\infty)(-2) \xrightarrow{\sim} \frac{H^{n-1}(\widetilde{S})(-1)}{\mathrm{Ker}(\delta_{n+1})},$$

which shows that $\mathrm{Coker}(\alpha_{n-1}) \cong \mathrm{Ker}(\delta_{n+1})(+1)$. We also have a short exact sequence

$$0 \longrightarrow H^n(\mathbb{P}_k^n \setminus \widetilde{S}) \longrightarrow H^{n-1}(\widetilde{S})(-1) \xrightarrow{\delta_{n+1}} H^{n+1}(\mathbb{P}_k^n) \longrightarrow 0$$

This yields the required isomorphism

$$\mathrm{Coker}(\alpha_{n-1}) \xrightarrow{\sim} \mathrm{Ker}(\delta_{n+1})(+1) \cong H^n(\mathbb{P}_k^n \setminus \widetilde{S})(+1).$$

Since δ_{n+2} is an isomorphism and $H^{n+2}(\mathbb{P}_k^n) = 0$ it follows that $H^n(\widetilde{S})(-1) = 0$. By looking at the diagram (3.5.4) we find that $\mathrm{Ker}(\alpha_n) \cong H^{n-2}(\widetilde{S}_\infty)(-1)$. But since $n-1$ is even we know from proposition 3.5.1 that the Gysin map $H^{n-1}(\mathbb{P}_k^{n-1} \setminus \widetilde{S}_\infty) \to H^{n-2}(\widetilde{S}_\infty)(-1)$ is an isomorphism. This gives an isomorphism

$$\mathrm{Ker}(\alpha_n) \xrightarrow{\sim} H^{n-1}(\mathbb{P}_k^{n-1} \setminus \widetilde{S}_\infty).$$

□

Proposition 3.5.4. *The short exact sequence (3.5.5) is compatible with the actions of the groups $\langle \zeta_d \rangle$ and $G(\underline{w})$.*

Proof. This is a result of the following property: any Cartesian square

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

with $Z' \subset X'$ and $Z \subset X$ closed immersions of codimension c gives rise to a morphism

$$\begin{array}{ccccccc} \dots \longrightarrow & H^i_{rig}(X \setminus Z) & \longrightarrow & H^{i+1-2c}_{rig}(Z)(-c) & \longrightarrow & H^{i+1}_{rig}(X) & \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ \dots \to & H^i_{rig}(X' \setminus Z') & \to & H^{i+1-2c}_{rig}(Z')(-c) & \to & H^{i+1}_{rig}(X') & \to \dots \end{array}$$

between the Gysin sequences. To prove the proposition we apply this claim to the actions of $\langle\zeta_d\rangle$ and $G(\underline{w})$ and their restrictions to closed subschemes. In this way, one sees that all the Gysin sequences appearing in the proof of proposition 3.5.3 are equivariant w.r.t. the groups $\langle\zeta_d\rangle$ and $G(\underline{w})$. □

3.5.2 Proof of the homogeneous case

In this paragraph we prove an analogue of proposition 3.4.1 for the rigid cohomology of smooth projective hypersurfaces (i.e. in the homogeneous case $\underline{w} = (1, \ldots, 1)$). This is sufficient to prove theorem 3.1.11 for homogeneous singularities, similarly to the proof of proposition 3.4.3. The starting point is to look at the monodromy-invariants of the sequence (3.5.5).

Proposition 3.5.5. *Let A, B and C be vector spaces over a field K of characteristic zero with an action of a finite group G and consider a G-equivariant short exact sequence*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

Then the induced sequence of maps

$$0 \longrightarrow A^G \xrightarrow{\alpha} B^G \xrightarrow{\beta} C^G \longrightarrow 0$$

is still exact.

Proof. We need to show that $\mathrm{Ker}(\beta) \subset \mathrm{Im}(\alpha)$ and that β is surjective. For the second claim, choose an element $c = \beta(b) \in C^G$ with $b \in B$. Then

$$c = \frac{1}{|G|} \sum_{g \in G} \beta(g \cdot b) = \beta\left(\frac{1}{|G|} \sum_{g \in G} g \cdot b\right)$$

and $\frac{1}{|G|} \sum_{g \in G} g \cdot b \in B^G$. This shows that the restriction $\beta\colon B^G \to C^G$ is surjective. The proof of the first claim is similar. □

From this lemma we obtain a short exact sequence

$$0 \longrightarrow H^n(\mathbb{P}^n_k \setminus \widetilde{S})(+1)^{\langle\zeta_d\rangle} \longrightarrow H^{n-1}(\widetilde{S} \setminus \widetilde{S}_\infty)^{\langle\zeta_d\rangle} \longrightarrow H^{n-1}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{\langle\zeta_d\rangle} \longrightarrow 0 \tag{3.5.6}$$

Next we show that the cohomology space $H^n(\mathbb{P}^n_k \setminus \widetilde{S})$ has no monodromy invariants.

Proposition 3.5.6. *The subspace of $H^n(\mathbb{P}^n_k \setminus \widetilde{S})$ that is fixed by $\langle\zeta_d\rangle$ is trivial.*

Proof. Let ϕ_{ζ_d} denote the map on $H^n(\mathbb{P}^n_k \setminus \widetilde{S})$ that comes from ζ_d. It suffices to show that the linear map $\phi_{\zeta_d} - \mathrm{Id}$ is invertible.

We fix a homogeneous lift $\widetilde{\mathcal{G}} \in \mathcal{V}[x_1, \ldots, x_n]$ of the polynomial $\widetilde{g}$. We have explained in paragraph 3.4.2 that $\widetilde{\mathcal{G}}$ can be chosen in such a way that the monodromy action lifts to the hypersurface $\widetilde{\mathcal{S}} = Z(\widetilde{\mathcal{G}} - x_0^d) \subset \mathbb{P}^n_{\mathcal{V}}$. More specifically: the unit root ζ_d in definition 3.3.4 needs to be replaced by its Teichmüller lift η_d. The Baldassarri-Chiarellotto theorem then gives us a monodromy-equivariant isomorphism of K-vector spaces

$$H^n_{dR}(\mathbb{P}^n_K \setminus \widetilde{\mathcal{S}}_K) \xrightarrow{\sim} H^n_{rig}(\mathbb{P}^n_k \setminus \widetilde{S}).$$

Therefore $\phi_{\zeta_d} - \mathrm{Id}$ can be interpreted as a map on the space $H^n_{dR}(\mathbb{P}^n_K \setminus \widetilde{\mathcal{S}}_K)$. We can use the theory of Griffiths to describe this map more explicitly.

For $h = 1, \ldots, n$ and $i = 0, \ldots, d-2$, find a set $\mathcal{B}_{h,i}$ of monomials in $K[x_1, \ldots, x_n]$ of degree $hd - n - 1 - i$ whose classes modulo the Jacobian ideal $J = (\partial_1 \widetilde{\mathcal{G}}, \ldots, \partial_n \widetilde{\mathcal{G}})$ form a basis of the homogeneous part

$$\left(\frac{K[x_1, \ldots, x_n]}{J} \right)_{hd-n-1-i}.$$

Then we claim that the set

$$\mathcal{B}_h = \bigcup_{i=0}^{d-2} x_0^i \cdot \mathcal{B}_{h,i} \tag{3.5.7}$$

forms a basis for the space

$$\left(\frac{K[x_0, x_1, \ldots, x_n]}{(\partial_0(\widetilde{\mathcal{G}} - x_0^d), \partial_1(\widetilde{\mathcal{G}} - x_0^d), \ldots, \partial_n(\widetilde{\mathcal{G}} - x_0^d))} \right)_{hd-n-1}. \tag{3.5.8}$$

If one rewrites the Jacobian ideal of the polynomial $\widetilde{\mathcal{G}} - x_0^d$ as

$$J \cdot K[x_0, x_1, \ldots, x_n] + (x_0^{d-1})$$

then it is easy to see that (3.5.7) is indeed a generating set. Also, the Hilbert function $\mathrm{Hilb}(\beta)$ of the Jacobi ring of $\widetilde{\mathcal{G}} - x_0^d$ is given by

$$\mathrm{Hilb}(\beta) = \sum_{i=0}^{d-2} \dim \left(\frac{K[x_1, \ldots, x_n]}{J} \right)_{\beta - i}.$$

This identity can be proved using the techniques from paragraph 4.1 in the next chapter. It follows that (3.5.7) has the correct number of elements, so it forms a basis for the space (3.5.8). According to proposition 1.2.4 we may now choose

$$\bigcup_{h=1}^{n} \left\{ \frac{m\,\Omega}{(\widetilde{\mathcal{G}} - x_0^d)^h} \mid m \in \mathcal{B}_h \right\}$$

as a basis for $H^n_{dR}(\mathbb{P}^n_K \setminus \widetilde{\mathcal{S}}_K)$. It is easy to express ϕ_{ζ_d} in terms of this basis:

$$\phi_{\zeta_d}\left(\frac{m\,\Omega}{(\widetilde{\mathcal{G}} - x_0^d)^h}\right) = \eta_d^{(hd-n-1-i)+n}\frac{m\,\Omega}{(\widetilde{\mathcal{G}} - x_0^d)^h}$$

if $m \in \mathcal{B}_{h,i}$. That is, the matrix of ϕ_{ζ_d} is diagonal w.r.t. this basis. In order to prove that $\phi_{\zeta_d} - \mathrm{Id}$ is invertible it suffices to show that $\eta_d^{(hd-n-1-i)+n} - 1 \neq 0$ for $i = 0, \ldots, d-2$. But this is is equivalent to saying that $(i+1) \not\equiv 0 \bmod d$ for $i = 0, \ldots, d-2$, which is trivially true. □

Proposition 3.5.7. *There is a canonical Frobenius-equivariant isomorphism*

$$H^{n-1}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty) \xrightarrow{\sim} H^{n-1}(\widetilde{S} \setminus \widetilde{S}_\infty)^{\langle \zeta_d \rangle}.$$

This isomorphism is also $G(\underline{w})$-equivariant.

Proof. Recall that the entire cohomology space $H^{n-1}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$ is monodromy-invariant. The proposition then follows immediately from the short exact sequence (3.5.6) and the fact that $H^n(\mathbb{P}^n_k \setminus \widetilde{S})^{\langle \zeta_d \rangle} = 0$. □

Remark 3.5.8. Proposition 3.5.7 can also be proved in a shorter way. Indeed, the scheme $\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty$ may be interpreted as the quotient of $\widetilde{S} \setminus \widetilde{S}_\infty$ under the action of $\langle \zeta_d \rangle$. On the level of geometric points, the quotient map is given by

$$(a_0 \colon a_1 \colon \ldots \colon a_n) \mapsto \left(\frac{a_1}{a_0} \colon \ldots \colon \frac{a_n}{a_0}\right).$$

The $\langle \zeta_d \rangle$-action on $\widetilde{S} \setminus \widetilde{S}_\infty$ is obviously free, since it is generated by

$$\phi_{\zeta_d} \colon (a_0 \colon a_1 \colon \ldots \colon a_n) \mapsto (\zeta_d^{-1}\, a_0 \colon a_1 \colon \ldots \colon a_n)$$

and $a_0 \neq 0$ by construction. We can then apply proposition 3.3.6. The advantage of the longer proof of proposition 3.5.7 is that it is as close as possible to the proof of proposition 3.4.1. In other words: we have shown that the theory of Steenbrink-Dolgachev-Dimca also holds for the rigid cohomology of smooth projective hypersurfaces.

3.5.3 Proof of the general case

We are now ready to prove theorem 3.1.11. We start with a lemma about the rigid cohomology of $\mathbb{A}^1_k \setminus \{0\}$.

Proposition 3.5.9. *We have $H^i(\mathbb{A}^1_k \setminus \{0\}) = 0$ for $i \notin \{0, 1\}$. The spaces $H^i(\mathbb{A}^1_k \setminus \{0\})$ for $i = 0, 1$ are both one-dimensional. Frobenius acts as the identity on $H^0(\mathbb{A}^1_k \setminus \{0\})$ and as $q\sigma$ on $H^1(\mathbb{A}^1_k \setminus \{0\})$. The action of $\langle \zeta_d \rangle$ on $H^\bullet(\mathbb{A}^1_k \setminus \{0\})$ that is induced by the action of $\langle \zeta_d \rangle$ on $\mathbb{A}^1_k \setminus \{0\}$ (see paragraph 3.3.3) is trivial.*

Proof. The first claim follows from the fact that $\mathbb{A}^1_k \setminus \{0\}$ is smooth affine of dimension 1. It is moreover clear that $H^0(\mathbb{A}^1_k \setminus \{0\})$ is one-dimensional and that $\langle \zeta_d \rangle$ acts as the identity on $H^0(\mathbb{A}^1_k \setminus \{0\})$. By considering the Gysin sequence for $\{0\} \subset \mathbb{A}^1_k$ it is easy to see that $H^1(\mathbb{A}^1_k \setminus \{0\})$ is one-dimensional with Frobenius acting as $q\sigma$.

In order to study the action of $\langle \zeta_d \rangle$ on $H^1(\mathbb{A}^1_k \setminus \{0\})$ we write $\mathbb{A}^1_k \setminus \{0\} = \mathbb{P}^1_k \setminus \{0, \infty\}$. We use the coordinates z_0, z_1 on $\mathbb{P}^1_k$. According to theorem 1.2.2 (Baldassarri-Chiarellotto) we have an isomorphism of K-vector spaces

$$H^1_{dR}(\mathbb{P}^1_K \setminus Z(z_0 z_1)) \xrightarrow{\sim} H^1(\mathbb{P}^1_k \setminus Z(z_0 z_1)).$$

Under this identification the action of $\langle \zeta_d \rangle$ on $H^1(\mathbb{P}^1_k \setminus Z(z_0 z_1))$ corresponds to the action on differential forms that is generated by the rule

$$\begin{cases} z_0 & \mapsto & z_0 \\ z_1 & \mapsto & \eta_d^{-1} z_1 \end{cases}$$

with $\eta_d \in \mathcal{V}$ denoting the Teichmüller lift of $\zeta_d \in k$.

By the theory of Griffiths (see proposition 1.2.4) we know that the cohomology space $H^1_{dR}(\mathbb{P}^1_K \setminus Z(z_0 z_1))$ is generated by the differential form

$$\frac{z_0 \cdot dz_1 - z_1 \cdot dz_0}{z_0 z_1}.$$

One sees immediately that this form is invariant under $\langle \zeta_d \rangle$. This proves that $\langle \zeta_d \rangle$ acts as the identity on $H^1(\mathbb{A}^1_k \setminus \{0\})$. □

The next step is to apply the Künneth formula to determine the $\langle \zeta_d \rangle$-invariant cohomology of $(S \setminus S_\infty) \times (\mathbb{A}^1_k \setminus \{0\})$.

Proposition 3.5.10. *Consider the action of $\langle \zeta_d \rangle$ on the product*

$$(S \setminus S_\infty) \times (\mathbb{A}^1_k \setminus \{0\}),$$

as defined in paragraph 3.3.3. Also assume that $n \geq 3$. Then the following properties hold:

i) We have

$$H^i((S \setminus S_\infty) \times (\mathbb{A}^1_k \setminus \{0\}))^{\langle \zeta_d \rangle} = 0$$

for $i \notin \{0, n-1, n\}$.

ii) There is a Frobenius-equivariant isomorphism

$$H^{n-1}((S \setminus S_\infty) \times (\mathbb{A}^1_k \setminus \{0\}))^{\langle \zeta_d \rangle} \xrightarrow{\sim} H^{n-1}(S \setminus S_\infty)^{\langle \zeta_d \rangle}.$$

iii) There is a Frobenius-equivariant isomorphism

$$H^n((S \setminus S_\infty) \times (\mathbb{A}^1_k \setminus \{0\}))^{\langle\zeta_d\rangle} \xrightarrow{\sim} H^{n-1}(S \setminus S_\infty)^{\langle\zeta_d\rangle}(-1).$$

Proof. By the Künneth formula (1.2.21) and proposition 3.5.9 we have a Frobenius-equivariant isomorphism for every $i \geq 0$:

$$H^i((S \setminus S_\infty) \times (\mathbb{A}^1_k \setminus \{0\})) \xrightarrow{\sim} \bigoplus_{j=0}^{1} H^{i-j}(S \setminus S_\infty) \otimes H^j(\mathbb{A}^1_k \setminus \{0\}). \quad (3.5.9)$$

Now observe that $H^i(S \setminus S_\infty) = 0$ for $i \notin \{0, n-1\}$. This is a consequence of the first point of proposition 3.3.6. Indeed, the affine Milnor fibers $S \setminus S_\infty$ and $\widetilde{S} \setminus \widetilde{S}_\infty$ are smooth affine, and (after a finite base field extension) we have an identification $S \setminus S_\infty \cong (\widetilde{S} \setminus \widetilde{S}_\infty)/G(\underline{w})$. We know from proposition 3.5.2 that $H^i(\widetilde{S} \setminus \widetilde{S}_\infty) = 0$ for $i \notin \{0, n-1\}$. The same property then holds for the cohomology of $S \setminus S_\infty$.

Point i) follows immediately from this observation.

To prove points ii) and iii) it remains to show that the isomorphisms (3.5.9) for $i = n-1$ and $i = n$ are compatible with the action of $\langle\zeta_d\rangle$. Let ϕ_1 resp. ϕ_2 denote the automorphisms on $S \setminus S_\infty$ resp. on $\mathbb{A}^1_k \setminus \{0\}$ that come from the unit root ζ_d. The action of $\langle\zeta_d\rangle$ on the product is then given by $\phi_1 \times \phi_2$. Under the Künneth formula, the induced map $H^i(\phi_1 \times \phi_2)$ on the left-hand side of equation (3.5.9) corresponds to sum of the tensor products $H^{i-j}(\phi_1) \otimes H^j(\phi_2)$. But we know from proposition 3.5.9 that $H^j(\phi_2)$ is the identity map. For $i = n-1$ and $i = n$ there is only one nonzero term on the right-hand side of equation (3.5.9), and we obtain the isomorphisms ii) and iii) by considering the eigenspaces at eigenvalue 1 on both sides of the equation. □

Now we can combine our results from all the previous sections.

Proof of Theorem 3.1.11. By using propositions 3.2.1 and 3.3.10 and the first point of proposition 3.5.10 we see that $H^i_{\{0\}}(Y) = 0$ for $i \notin \{n-1, n, 2n-2\}$. This proves point iii) of the theorem. In a similar way, using the last two points of proposition 3.5.10, we obtain Frobenius-equivariant isomorphisms

$$H^{n-1}_{\{0\}}(Y) \xrightarrow{\sim} H^{n-1}(S \setminus S_\infty)^{\langle\zeta_d\rangle}(+1)$$

and

$$H^n_{\{0\}}(Y) \xrightarrow{\sim} H^{n-1}(S \setminus S_\infty)^{\langle\zeta_d\rangle}. \quad (3.5.10)$$

The second point of the theorem follows from these two identifications. Next we compose (3.5.10) with the isomorphism

$$H^{n-1}(S \setminus S_\infty)^{\langle\zeta_d\rangle} \xrightarrow{\sim} \left(H^{n-1}(\widetilde{S} \setminus \widetilde{S}_\infty)^{\langle\zeta_d\rangle}\right)^{G(\underline{w})}$$

of proposition 3.4.5. By applying proposition 3.5.7 we obtain a Frobenius-equivariant isomorphism

$$H^n_{\{0\}}(Y) \xrightarrow{\sim} H^{n-1}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}.$$

This concludes the proof of the theorem. $\square$

We end this chapter with a couple of remarks.

Remark 3.5.11. The statement of theorem 3.1.11 needs to be only slightly modified for the case $n = 2$. Instead of the isomorphism from point ii) of proposition 3.2.1 we have a Frobenius-equivariant short exact sequence

$$0 \longrightarrow H^2(\mathbb{A}^2_k \setminus Y)(+1) \longrightarrow H^2_{\{0\}}(Y) \longrightarrow H^0(\{0\})(-1) \longrightarrow 0$$

By the same arguments as above we still have an isomorphism

$$H^2(\mathbb{A}^2_k \setminus Y)(+1) \xrightarrow{\sim} H^1(\mathbb{P}^1_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}.$$

This information is sufficient if one is only interested in the dimension or in the characteristic polynomial of Frobenius (see chapter 4).

Remark 3.5.12. We do not know if the proof of theorem 3.1.11 can be adapted for étale cohomology. Most of the techniques that we used in this chapter do carry over to étale cohomology. Note in particular that the commuting of $H^\bullet_{\acute{e}t}(\_\_, \mathbb{Q}_\ell)$ with finite étale Galois covers is a special case of the *Hochschild-Serre spectral sequence*. See [Mil80, Theorem III.2.20] for details. However, the proofs of propositions 3.4.5 and 3.5.6 rely on the relation between rigid cohomology and the de Rham cohomology of a lift. This technique is typical for p-adic cohomology.

Chapter 4

Computation of invariants

In this chapter we will use the isomorphism

$$H^n_{rig,\{0\}}(Y) \xrightarrow{\sim} H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})} \tag{4.0.1}$$

from theorem 3.1.11 to compute some invariants of a weighted homogeneous hypersurface singularity. We have already explained that the abstract cohomology space $H^n_{rig,\{0\}}(Y)$ is not amenable to direct computation, while the Frobenius action on $H^{n-1}_{rig}(\mathbb{P}^{n-1} \setminus \widetilde{S}_\infty)^{G(\underline{w})}$ can "almost" be computed with the AKR algorithm. The most important result of this chapter is that the "almost" restriction can be lifted. Indeed, in section 4.2 we will prove some results that allow us to formulate a modified version of the AKR algorithm.

Throughout this chapter we keep the notations from theorem 3.1.11. In particular we assume that our singularity $Y = Z_{\mathbb{A}^n_k}(g)$ satisfies all the assumptions from definition 3.1.7.

We work over a finite ground field $k = \mathbb{F}_q$ with q a power of a prime p. We will only consider the q-power Frobenius on this base field. As usual we take $\mathcal{V}$ to be the ring $W(k)$ of Witt vectors over k and $K = \operatorname{Frac} W(k)$.

For a given a weighted homogeneous polynomial $g \in k[x_1, \ldots, x_n]$ we denote by $\mathcal{G} \in \mathcal{V}[x_1, \ldots, x_n]$ a weighted homogeneous lift that is obtained by lifting the nonzero coefficients of g. This also allows to define lifts $\mathcal{S}_\infty$, $\mathcal{S}$, $\widetilde{\mathcal{S}}_\infty$, $\widetilde{\mathcal{S}}$, ... together with their usual actions of the group $G(\underline{w})$. We have already carried out this construction in more detail in paragraph 3.4.2.

An important remark for the rest of this chapter is that the cohomology spaces $H^\bullet_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$ have a concrete description in terms of differential forms. Indeed, the hypersurface $\widetilde{S}_\infty = Z(\widetilde{g}) \subset \mathbb{P}^{n-1}_k$ is smooth by assumption and the theorem of Baldassarri-Chiarellotto (see theorem 1.2.2) gives us an isomorphism of K-vector spaces

$$H^\bullet_{dR}(\mathbb{P}^{n-1}_K \setminus (\widetilde{\mathcal{S}}_K)_\infty) \xrightarrow{\sim} H^\bullet_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty). \tag{4.0.2}$$

By the work of Griffiths the space on the left-hand side can be explicitly described in terms of differential forms. See the explanations in the introductory paragraph 1.2.3. The $G(\underline{w})$-action on $H^{\bullet}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$ can now be understood in terms of a $G(\underline{w})$-action on differential forms. At least after a finite base field extension of k to make sure that the field K contains the necessary unit roots $\zeta_{w_1}, \ldots, \zeta_{w_n}$. In this chapter we say that a differential form is $G(\underline{w})$-invariant if this is the case over some (and hence over *any*) suitable field extension. In a similar way we can talk about the $G(\underline{w})$-invariant de Rham classes in $H^{\bullet}_{dR}(\mathbb{P}^{n-1}_K \setminus (\widetilde{\mathcal{S}}_K)_\infty)$. Under the identification (4.0.2) these classes precisely correspond to the elements in $H^{\bullet}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}$.

We start this chapter by studying the dimension of local rigid cohomology, which is an easy invariant. In section 4.2 we discuss a modification of the AKR algorithm that allows us to approximate the Frobenius on the local cohomology. From this algorithm we then derive a more refined invariant. The final section 4.3 contains some examples of weighted homogeneous singularities. For some of these examples we are able to exactly determine the Frobenius action.

4.1 The dimension of local cohomology

In this section we derive a formula for the dimension of the local rigid cohomology of a weighted homogeneous singularity. It follows from theorem 2.1.1 that this is an invariant of the local cohomology. For simplicity we will limit the discussion to the case $n \geq 3$, so that theorem 3.1.11 is applicable. The results in this paragraph can easily be adapted to the case $n = 2$ by making use of remark 3.5.11.

After choosing an embedding $K \hookrightarrow \mathbb{C}$ and applying Grothendieck's theorem [Gro66, Theorem 1] to the identification (4.0.2) we obtain an equality

$$\dim_K H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})} = \dim_{\mathbb{C}} H^{n-1}(\mathbb{P}^{n-1}_{\mathbb{C}} \setminus (\widetilde{\mathcal{S}}_{\mathbb{C}})_\infty,\, \mathbb{C})^{G(\underline{w})}.$$

By proposition 3.4.1 we also have an equality

$$\dim_{\mathbb{C}} H^{n-1}(\mathbb{P}^{n-1}_{\mathbb{C}} \setminus (\widetilde{\mathcal{S}}_{\mathbb{C}})_\infty,\, \mathbb{C})^{G(\underline{w})} = \dim_{\mathbb{C}} H^{n-1}(\mathbb{P}_{\mathbb{C}}(\underline{w}) \setminus (\mathcal{S}_{\mathbb{C}})_\infty,\, \mathbb{C}).$$

Therefore it suffices to consider the dimension of a quasi-smooth complex weighted homogeneous hypersurface. These objects are well studied. In particular there is the following result.

Proposition 4.1.1. *Let $A = \mathbb{C}[x_1, \ldots, x_n]$ denote the polynomial ring with the grading $\deg x_i = w_i$. Also choose a weighted homogeneous polynomial $P \in A$ of degree d such that the weighted projective hypersurface $V = Z(P) \subset \mathbb{P}_{\mathbb{C}}(\underline{w})$ is quasi-smooth. Consider the homogeneous ideal $I = (\partial_1 P, \ldots, \partial_n P) \subset A$ and let $h_{A/I}$ denote the Hilbert function of the module A/I. Then the highest Betti* *number of the complement $\mathbb{P}_{\mathbb{C}}(\underline{w}) \setminus V$ is given by the formula*

$$\dim H^{n-1}(\mathbb{P}_{\mathbb{C}}(\underline{w}) \setminus V, \mathbb{C}) = \sum_{\alpha=1}^{n-1} h_{A/I}\left(\alpha d - \sum_{i=1}^{n} w_i\right). \tag{4.1.1}$$

Proof. This follows from [Dol82, Theorem 4.3.2], combined with the Hodge decomposition from equation (3.4.2). □

It remains to compute the Hilbert function on the right-hand side of (4.1.1). The assumption that V is quasi-smooth makes this particularly easy. Indeed, assume that M is any finitely generated graded A-module with resolution

$$\ldots \longrightarrow \bigoplus_j A(-j)^{\beta_{1j}} \longrightarrow \bigoplus_j A(-j)^{\beta_{0j}} \longrightarrow M \longrightarrow 0$$

Then the Hilbert-Poincaré series of M is given by

$$\psi_M(t) = \frac{\sum_{i,j}(-1)^i \beta_{ij} t^j}{\prod_i (1 - t^{w_i})}. \tag{4.1.2}$$

Also see [Eis05, Theorem 1.11]. The assumption that V is quasi-smooth precisely means that the affine cone is smooth outside the origin. By using the Euler relation

$$dP = \sum_{i=1}^{n} w_i x_i \, \partial_i P$$

we see that $Z_{\mathbb{A}^n_{\mathbb{C}}}(I)$ is the origin of $\mathbb{A}^n_{\mathbb{C}}$. In other words, the polynomials $\partial_i P$ form a regular sequence in A (see proposition 4.1.5 below). Since $\partial_i P$ is weighted homogeneous of degree $d - w_i$, the Koszul complex gives us a graded resolution

$$\ldots \longrightarrow F_i \longrightarrow \ldots \longrightarrow \bigoplus_{l=1}^{n} A(-(d - w_l)) \longrightarrow A \longrightarrow \tfrac{A}{I} \longrightarrow 0$$

where

$$F_i = \bigoplus_{l_1=1}^{n} \bigoplus_{l_2=l_1}^{n} \cdots \bigoplus_{l_i=l_{i-1}}^{n} A(-(d - w_{l_1}) - (d - w_{l_2}) - \ldots - (d - w_{l_i})).$$

So for the β_{ij} in equation (4.1.2) we obtain:

$$\beta_{ij} = \left| \left\{ (l_1, \ldots, l_i) \mid l_1 < \ldots < l_i \text{ and } id - \sum_{\alpha=1}^{i} w_{l_\alpha} = j \right\} \right|.$$

For these values β_{ij} we have the identity

$$\sum_{i=1}^{n} (-1)^i \sum_{j \in \mathbb{Z}} \beta_{ij} t^j = \prod_{i=1}^{n} (1 - t^{d - w_i})$$

and it follows that

$$\psi_{A/I}(t) = \prod_{i=1}^{n} \frac{t^{d-w_i} - 1}{t^{w_i} - 1}. \tag{4.1.3}$$

So in order to compute the right-hand side of equation (4.1.1) it suffices to determine the relevant coefficients of the power series (4.1.3).

Using the formula (4.1.3) it is easy to see that $\psi_{A/I}(t)$ is in fact a polynomial. Indeed, the series $\psi_{A/I}(t)$ converges for $|t| < 1$. Since the coefficients of $\psi_{A/I}(t)$ belong to $\mathbb{N}$, the limit for $t \to 1^-$ is either a positive number or $+\infty$. In the first case $\psi_{A/I}(t)$ must be a polynomial, and $\lim_{t\to 1^-} \psi_{A/I}(t) = \psi_{A/I}(1)$ is the sum of the coefficients. In our situation the limit for $t \to 1^-$ is easy to determine. Using the rule of de L'Hôpital we find

$$\lim_{t\to 1^-} \prod_{i=1}^{n} \frac{t^{d-w_i} - 1}{t^{w_i} - 1} = \prod_{i=1}^{n} \left(\frac{d}{w_i} - 1 \right).$$

This provides an upper bound

$$\sum_{l=0}^{\infty} h_{A/I}(l) = \psi_{A/I}(1) = \prod_{i=1}^{n} \left(\frac{d}{w_i} - 1 \right).$$

for the right-hand side of (4.1.1).

By looking at the degree of the polynomial $\psi_{A/I}(t)$ we also obtain the following fact:

$$h_{A/I}(l) = 0 \quad \text{for} \quad l > nd - 2 \cdot \sum_{i=1}^{n} w_i. \tag{4.1.4}$$

For completeness we write down our computations in the form of a proposition.

Proposition 4.1.2. *Consider a weighted homogeneous singularity $Y = Z_{\mathbb{A}^n_k}(g)$ with $g \in k[x_1, \ldots, x_n]$ a weighted homogeneous polynomial of degree d w.r.t. weights $(w_1, \ldots, w_n)$ that satisfies all the assumptions of theorem 3.1.11. Then the dimension of the local rigid cohomology is given by the formula*

$$\dim H^n_{rig,\{0\}}(Y) = \sum_{\alpha=1}^{n-1} c_{\alpha d - s}$$

where $s = \sum_{i=1}^{n} w_i$ and c_j is the coefficient of t^j in the power series

$$\prod_{i=1}^{n} \frac{t^{d-w_i} - 1}{t^{w_i} - 1}.$$

The dimension is bounded from above by the Milnor number

$$\mu = \prod_{i=1}^{n} \left(\frac{d}{w_i} - 1 \right).$$

Remark 4.1.3. The formulas for the Hilbert-Poincaré series and the Milnor number, as well as the statement of proposition 4.1.5 below, are already well-known. We wrote down their derivations to make it clear that the same calculations work over any base field whose characteristic does not divide $\deg_{\underline{w}} P$. This is relevant for the proof of proposition 4.2.3 in the next section.

Remark 4.1.4. In the case of a homogeneous singularity (i.e. $\widetilde{g} = g$) there is another result that can be used to compute the dimension of the local cohomology. In this case we have an equality

$$\dim H^n_{rig,\{0\}}(Y) = \dim H^{n-1}(\mathbb{P}^{n-1}_{\mathbb{C}} \setminus (\widetilde{\mathcal{S}}_{\mathbb{C}})_\infty, \mathbb{C})$$

and we may use the formula

$$\dim H^{n-1}(\mathbb{P}^{n-1}_{\mathbb{C}} \setminus (\widetilde{\mathcal{S}}_{\mathbb{C}})_\infty, \mathbb{C}) = \frac{d-1}{d}((d-1)^{n-1} + (-1)^n).$$

For an analytical proof of this formula see [Dim92, Exercise 5.3.7]. An algebraic proof can be found in [Mon70, Theorem 8.3].

The only thing that is left to do is to justify the earlier claim that the partial derivatives $\partial_1 P, \ldots, \partial_n P$ form a regular sequence. We show that this property is true over any base field, not only over $\mathbb{C}$.

Proposition 4.1.5. *Let k be any field and consider the polynomial ring $A = k[x_1, \ldots, x_n]$ with the grading $\deg x_i = w_i$. Let $f_1, \ldots, f_n \in A$ be (weighted) homogeneous polynomials such that the system of equations*

$$\begin{cases} f_1(a_1, \ldots, a_n) &= 0 \\ &\vdots \\ f_n(a_1, \ldots, a_n) &= 0 \end{cases}$$

has no solutions for $(a_1, \ldots, a_n) \in (\overline{k})^n$, except for $\underline{a} = \underline{0}$. Then $f_1, \ldots, f_n$ form a regular sequence in A.

Proof. Write $I = (f_1, \ldots, f_n)$. It then follows from the Nullstellensatz that

$$\sqrt{I \cdot \overline{k}[x_1, \ldots, x_n]} = (x_1, \ldots, x_n) \cdot \overline{k}[x_1, \ldots, x_n].$$

This means that for every x_i, there exists a $d_i > 0$ such that

$$x_i^{d_i} \in I \cdot \overline{k}[x_1, \ldots, x_n].$$

But since also $x_i^d \in A$ we find that $x_i^{d_i} \in I$. This results in an integer $d > 0$ such that

$$(x_1, \ldots, x_n)^d \subset I.$$

This means that the f_i form a system of homogeneous parameters for A. See chapter III in [NVO79] for the precise definition. Since the ring A is Cohen-Macaulay, it follows from [NVO79, Proposition III.3.9] that the f_i form a regular sequence in A. $\square$

This proposition implies that if a weighted homogeneous polynomial $P \in A$ defines an isolated singularity at the origin (i.e. the hypersurface $V = Z(P) \subset \mathbb{P}_k(\underline{w})$ is quasi-smooth) and if the characteristic of k does not divide $\deg_{\underline{w}} P$, then the partial derivatives $\partial_1 P, \ldots, \partial_n P$ form a regular sequence.

4.2 Approximating Frobenius with a modified AKR algorithm

In this section we describe how the algorithm from [AKR11] can be modified to approximate the Frobenius action on $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}$ for $n \geq 2$, starting from the identification (4.0.2).

The theoretical background for this algorithm is largely the same as in [AKR11]. We will cover this in paragraph 4.2.1.

In paragraphs 4.2.3 and 4.2.4 we show how the classical AKR algorithm can be rewritten to work on *weighted homogeneous* polynomials. The resulting algorithms 4.1 and 4.2 are concrete enough to implement. At the end of paragraph 4.2.4 we provide a link to our implementation in SAGE[1].

Paragraph 4.2.2 contains an overview of previously known variations of the AKR algorithm.

In paragraph 4.2.6 we give an in-depth comparison of our algorithms with some program code that is part of the *Frobenius project* by Johan de Jong [dJ06]. This project was originally intended to approximate the Frobenius action on the rigid cohomology of (the complement of) a quasi-smooth hypersurface in a weighted projective space. However, we believe that this code is computationally equivalent to our modification of the AKR algorithm, at least under our assumption that $\widetilde{S}_\infty$ is smooth. We argue that stronger theoretical foundations are needed in order to prove that the *Frobenius project* correctly approximates the Frobenius action on a quasi-smooth hypersurface.

In paragraph 4.2.7 we show that the modified AKR algorithm can be used to define another computable invariant of a weighted homogeneous hypersurface singularity. More specifically, we show how to compute the characteristic polynomial of Frobenius modulo a power of p.

[1] `http://www.sagemath.org`

4.2.1 Using AKR on $G(\underline{w})$-invariant forms

We start by verifying that the theoretical background for the AKR algorithm remains largely the same when one wishes to approximate the Frobenius on the $G(\underline{w})$-invariant part of cohomology.

Let R denote the polynomial ring $K[x_1, \dots, x_n]$ and consider the Jacobian ideal $J = (\partial_1 \widetilde{\mathcal{G}}, \dots \partial_n \widetilde{\mathcal{G}}) \subset R$.

Definition 4.2.1. Let D be an integer such that $D - n > 0$. Let $\mathcal{M}_D$ be a set of monomials of degree $D - n$ whose classes form a basis for the K-vector space $\left(\frac{R}{J}\right)_{D-n}$. Then we define $\mathcal{M}'_D$ to be the subset consisting of monomials $m = x_1^{a_1} \cdot \dots \cdot x_n^{a_n} \in \mathcal{M}_D$ that satisfy the following condition:

$$a_i \equiv -1 \bmod w_i \text{ for every } i \text{ such that } w_i > 1. \tag{4.2.1}$$

By convention we also define $\mathcal{M}_D = \mathcal{M}'_D = \emptyset$ for $D - n < 0$, $\mathcal{M}_n = \{1\}$ and

$$\mathcal{M}'_n = \begin{cases} 1 & \text{if } w_i = 1 \text{ for all } i \\ \emptyset & \text{otherwise} \end{cases}$$

Recall from the introductory paragraph 1.2.3 that for $d = \deg \widetilde{\mathcal{G}}$ and any choice of monomials $\{\mathcal{M}_{\alpha d}\}_{\alpha=1}^{n-1}$ as above the set

$$\bigcup_{\alpha=1}^{n-1} \left\{ \frac{m\,\Omega}{\widetilde{\mathcal{G}}^{\alpha}} \mid m \in \mathcal{M}_{\alpha d} \right\} \tag{4.2.2}$$

forms a basis for the de Rham cohomology space $H^{n-1}_{dR}(\mathbb{P}^{n-1}_K \setminus (\widetilde{\mathcal{S}}_K)_\infty)$. It is quite easy to show that the sets $\mathcal{M}'_{\alpha d}$ define a basis for the $G(\underline{w})$-invariant part of the de Rham cohomology.

Proposition 4.2.2. *For $\alpha \in \{1, \dots, n-1\}$ let $\mathcal{M}_{\alpha d}$ be a set of monomials of degree $\alpha d - n$ whose classes form a basis for $\left(\frac{R}{J}\right)_{\alpha d-n}$. Then the set*

$$\bigcup_{\alpha=1}^{n-1} \left\{ \frac{m\,\Omega}{\widetilde{\mathcal{G}}^{\alpha}} \mid m \in \mathcal{M}'_{\alpha d} \right\} \tag{4.2.3}$$

with $\mathcal{M}'_{\alpha d}$ as in definition 4.2.1 forms a basis for the $G(\underline{w})$-invariant de Rham cohomology

$$H^{n-1}_{dR}(\mathbb{P}^{n-1}_K \setminus (\widetilde{\mathcal{S}}_K)_\infty)^{G(\underline{w})}.$$

Proof. Recall that the $G(\underline{w})$-action on the de Rham cohomology comes from an action on differential forms. Since every element of $\mathcal{M}_{\alpha d}$ is a monomial it follows that the $G(\underline{w})$-action on differentials maps every element of (4.2.2) to a scalar multiple. Therefore the $G(\underline{w})$-invariant subset of (4.2.2) forms a basis for the $G(\underline{w})$-invariant de Rham cohomology.

For a given monomial m it is easy to see that a differential form

$$\frac{m\,\Omega}{\widetilde{\mathcal{G}}^t}$$

is $G(\underline{w})$-invariant if and only if $x_1 \dots x_n \cdot m$ is a monomial in $x_1^{w_1}, \dots, x_n^{w_n}$, using the fact that $G(\underline{w})$ acts the same on Ω as on the monomial $x_1 \cdot \ldots \cdot x_n$. But this is clearly equivalent to the condition (4.2.1). The proposition follows. □

In the rest of this paragraph we show that the basis (4.2.3) can be used together with the AKR algorithm.

First we prove that *any* basis (4.2.2) satisfies the conditions of [AKR11, Definition 3.4.2]. This ensures that all the properties in paragraphs 3.4 and 3.5 of [AKR11] hold, also with respect to the sub-basis (4.2.3). The proof relies on our assumption that the degree d is not divisible by $p = \mathrm{char}(k)$. See definition 3.1.7. It is sufficient to prove the following statement.

Proposition 4.2.3. *Let $\overline{J} = (\partial_1 \widetilde{g}, \dots, \partial_n \widetilde{g})$ be the reduction modulo p of the Jacobian ideal J of $\widetilde{\mathcal{G}}$. Then for every $\beta \geq 0$ we have*

$$\dim_k \left(\frac{k[x_1, \dots, x_n]}{\overline{J}} \right)_\beta = \dim_K \left(\frac{K[x_1, \dots, x_n]}{J} \right)_\beta \tag{4.2.4}$$

where the subscript denotes the homogeneous part of degree β.

Proof. Since we are assuming that $p \nmid d$ the Euler relation

$$d \cdot \widetilde{g} = \sum_{i=1}^{n} x_i \cdot \partial_i \widetilde{g}$$

is non-trivial. Our assumption that $\widetilde{S}_\infty$ is smooth then implies that the partial derivatives $\partial_1 \widetilde{g}, \dots, \partial_n \widetilde{g}$ form a regular sequence in $k[x_1, \dots, x_n]$. Therefore the Koszul complex forms a graded resolution for the module $\frac{k[x_1,\dots,x_n]}{\overline{J}}$. The left-hand side of (4.2.4) may then be computed as in section 4.1, using the same formal computations as for the right-hand side. The result follows. □

Next we verify that the Griffiths-Dwork reduction algorithm also works for the $G(\underline{w})$-invariant de Rham cohomology. For convenience we rewrite the basis (4.2.3) as

$$\left\{ \frac{m_j\,\Omega}{\widetilde{\mathcal{G}}^{\alpha_j}} \right\}_j$$

with j running over an appropriate index set and $m_j \in \mathcal{M}'_{\alpha_j \cdot d}$ for all j. If $\frac{A\,\Omega}{\widetilde{\mathcal{G}}^t}$ is a $G(\underline{w})$-invariant form then according to proposition 4.2.2 there is a unique linear combination

$$\frac{A\,\Omega}{\widetilde{\mathcal{G}}^t} = \sum_j \lambda_j \frac{m_j\,\Omega}{\widetilde{\mathcal{G}}^{\alpha_j}} \tag{4.2.5}$$

on the level of cohomology. But the basis (4.2.3) is a subset of (4.2.2), which is a basis for $H^{n-1}_{dR}(\mathbb{P}^{n-1}_K \setminus (\widetilde{\mathcal{S}}_K)_\infty)$. It follows that equation (4.2.5) is also the representation of $\frac{A\,\Omega}{\widetilde{\mathcal{G}}^t}$ in terms of this larger basis. As a consequence, the representation (4.2.5) may be computed using the classical Griffiths-Dwork reduction method that we have described in paragraph 1.2.3.

Finally, we verify that the basis (4.2.3) is compatible with the Frobenius lift that we discussed in paragraph 1.2.3. That is, if $\frac{m\,\Omega}{\widetilde{\mathcal{G}}^\alpha}$ is an element of the basis (4.2.3) then we would like that every term in the expansion

$$F\left(\frac{m\,\Omega}{\widetilde{\mathcal{G}}^\alpha}\right) = q^{n-1}\sum_{i\geq 0}\binom{\alpha+i-1}{i}\frac{x_1^{q-1}\cdot\ldots\cdot x_n^{q-1}\cdot F(m)\cdot(\widetilde{\mathcal{G}}^q - F(\widetilde{\mathcal{G}}))^i\,\Omega}{\widetilde{\mathcal{G}}^{q(i+\alpha)}} \tag{4.2.6}$$

is again a $G(\underline{w})$-invariant form. For this it suffices to verify that for every value of $i \geq 0$ the expression

$$x_1^q\cdot\ldots\cdot x_n^q\cdot F(m)\cdot(\widetilde{\mathcal{G}}^q - F(\widetilde{\mathcal{G}}))^i$$

is a polynomial in the variables $x_1^{w_1},\ldots,x_n^{w_n}$. First observe that

$$x_1^q\cdot\ldots\cdot x_n^q\cdot F(m) = F(x_1\cdot\ldots\cdot x_n\cdot m)$$

and that $x_1\cdot\ldots\cdot x_n\cdot m$ is a polynomial in $x_1^{w_1},\ldots,x_n^{w_n}$, since $m\in\mathcal{M}'_{\alpha d}$. The claim then follows immediately from the fact that for any polynomial P in the variables $x_1,\ldots,x_n$, we have

$$F(P(x_1^{w_1},\ldots,x_n^{w_n})) = P(x_1^{qw_1},\ldots,x_n^{qw_n}) = F(P)(x_1^{w_1},\ldots,x_n^{w_n}).$$

We now know that every *truncation* of the sum (4.2.6) is invariant under $G(\underline{w})$. Note that this does not automatically follow from the fact that the $G(\underline{w})$-invariant cohomology is Frobenius-stable.

In order to approximate the Frobenius on $H^{n-1}_{rig}(\mathbb{P}^{n-1}\setminus\widetilde{S}_\infty)^{G(\underline{w})}$ we can now truncate the sum (4.2.6) at the N^{th} term and use Griffiths-Dwork reduction to write the first N terms as a linear combination of our basis (4.2.3). If N is chosen big enough then the resulting matrix will approximate the Frobenius matrix with absolute precision r, for some value $r\geq 1$ that is chosen in advance. For a prescribed precision r a sufficiently large value for N can still be determined using the algorithm at the beginning of paragraph 3.5 in [AKR11][2]. However, the estimated value of N is likely to be too big. The reason is that the estimations in [AKR11] look at the entire Frobenius matrix on $H^{n-1}_{rig}(\mathbb{P}^{n-1}\setminus\widetilde{S}_\infty)$ and determine N such that the slowest converging entry in this matrix will have the required precision. Since the $G(\underline{w})$-invariant

[2] One should be careful that this algorithm actually computes a truncation level N that guarantees absolute precision r in the p-power Frobenius matrix. From this it is possible to derive precision bounds for the q-power Frobenius matrix. See paragraph 4.2.7 for details.

subspace usually has a much smaller dimension one can expect that the entries in the Frobenius matrix that correspond to this subspace converge faster. Undoubtedly it is possible to find better bounds for the Frobenius matrix of the $G(\underline{w})$-invariant cohomology, but we haven't investigated this.

4.2.2 Modifications of the AKR algorithm

In the previous paragraph we have shown that the Frobenius action on the $G(\underline{w})$-invariant cohomology $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}$ can be approximated using the standard AKR algorithm. More precisely: the Frobenius matrix on the $G(\underline{w})$-invariant cohomology can be seen as a submatrix of the Frobenius matrix on $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$.

Our goal in the rest of this section is to show that this clumsy "embedding" into the classical AKR algorithm is not necessary. We will prove that the algorithm of paragraph 4.2.1 can be implemented by only using operations on *weighted homogeneous* polynomials modulo the Jacobian ideal of $\mathcal{G}$.

In paragraph 4.2.3 we give an algorithm to compute a basis of the $G(\underline{w})$-invariant de Rham cohomology, using only the Jacobian ideal of $\mathcal{G}$. This is achieved by proving certain relations with the Jacobian ideal of $\widetilde{\mathcal{G}}$. In paragraph 4.2.4 we show how to carry out the reduction step of the AKR algorithm with respect to this basis. The reduction rules are completely written in terms of the Jacobian ideal of $\mathcal{G}$.

The results in paragraphs 4.2.3 and 4.2.4 yield an algorithm that we will refer to as the *modified AKR algorithm*. Unlike the classical AKR algorithm, our variation operates on *weighted homogeneous* polynomials. One should however remember that this algorithm is still computationally equivalent with the classical AKR algorithm, as described in paragraph 4.2.1. This does not mean that our modifications are trivial. Their correctness must be proved. Our most important result in this regard is proposition 4.2.8.

We emphasize that the modifications in paragraphs 4.2.3 and 4.2.4 only hold under the assumption that the degree $d = \deg \widetilde{\mathcal{G}} = \deg_{\underline{w}} \mathcal{G}$ is not divisible by the characteristic $p = \mathrm{char}(k)$. Indeed, in paragraph 4.2.1 we have shown that under this condition *any* basis (4.2.2) satisfies the conditions from [AKR11, Definition 3.4.2]. The basis that we will construct in paragraph 4.2.3 corresponds to a particular choice for the basis (4.2.2). When $p \mid d$ this basis can still be constructed, but there is no guarantee that it satisfies the conditions of [AKR11, Definition 3.4.2]. In fact, it seems that this basis is always bad when $p \mid d$.

The modified AKR algorithm also makes sense in the homogeneous case $\underline{w} = (1, \ldots, 1)$. In this way one obtains an alternative way to implement the classical AKR algorithm. This approach for the homogeneous case was already known to some people for at least a few years (the author learned it from his supervisor). However, the alternative implementation has only recently been published in [BLS13] and [Lai15]. These two papers contain (among other things) a generalization of the Griffiths-Dwork reduction for homogeneous equations that are not necessarily smooth. This goes in a different direction from our modification, which deals with weighted homogeneous equations that are quasi-smooth. The smooth homogeneous equations are precisely the intersection of these two classes, and in both cases one recovers the alternative implementation of the AKR algorithm. More precisely: the homogeneous cases of our algorithms 4.1 and 4.2 are implicit in the proof of [BLS13, Proposition 2]. They may also be seen as a special case of [Lai15, Algorithm 3], which is a generalization of Griffiths-Dwork reduction for homogeneous equations.

The original implementation of the AKR algorithm, which is available from the homepage of Kiran Kedlaya, also works for $p \mid d$. This implementation uses the more traditional Griffiths-Dwork reduction procedure that we have described in the introductory paragraph 1.2.3. The advantage of using the alternative implementation when $p \nmid d$ is that there is no need to do linear algebra in the quotient ring R/J. Indeed, algorithm 4.1 constructs a basis of the de Rham cohomology directly from a Gröbner basis of the Jacobian ideal. The coordinates of the cohomology class associated to a given differential form can then simply be read off from its Griffiths-Dwork reduction.

The algorithms 4.1 and 4.2 have previously been implemented by Johan de Jong, as part of his *Frobenius project*. However, we are not aware of any previous proofs for the correctness of these algorithms. Also, the aim of [dJ06] is to approximate the Frobenius on a weighted projective hypersurface. The explanations that come with the source code do not mention local cohomology at all. In paragraph 4.2.6 we will argue that the correctness of the code from [dJ06] relies on a certain conjectural relation between rigid cohomology objects.

Another remark about the implementation [dJ06] is that it should also work for $p \mid d$, according to the documentation. This suggests that our proofs related to algorithms 4.1 and 4.2 could be extended to deal with the case $p \mid d$.

We should also remark that there are several ways to speed up the classical AKR algorithm. Kloosterman [Klo08] has suggested to replace the Frobenius lift F by the Dworkian ψ, which is a left inverse. Costa [Cos15] has significantly improved the time and space complexity of the AKR algorithm, using the technique of *controlled reduction*. This result had previously been announced by Harvey [Har14]. It is possible that some of these improvements can be used together with our modifications.

4.2.3 Efficient construction of a $G(\underline{w})$-invariant basis

The naive way to compute the sets $\mathcal{M}'_D$ from the sets $\mathcal{M}_D$ by directly applying the definition (4.2.1) is generally not efficient. A striking example is provided in the code of the *Frobenius project* [dJ06]: consider the values $d = 64$ and $\underline{w} = (7, 8, 15, 19)$. In this case the basis (4.2.2) has cardinality 246141 while (4.2.3) has only 7 elements. In this paragraph we will give some results that allow us to efficiently compute a basis for the $G(\underline{w})$-invariant de Rham cohomology. The resulting algorithm 4.1 is also natural in our setting, since it only operates on the weighted homogeneous equation $\mathcal{G}$.

In this paragraph we heavily rely on the theory of Gröbner bases. We use chapter 21 of [vzGG13] as our reference. In particular we use $\preccurlyeq$ to denote a global monomial order. For a set of polynomials $\mathcal{P}$ we let $\mathrm{lm}(\mathcal{P})$ denote the set of leading monomials with respect to the chosen order $\preccurlyeq$. We may then also consider the ideal $\langle \mathrm{lm}(\mathcal{P}) \rangle$ that is generated by those leading monomials.

We start by fixing some notations.

- The ring $R = K[x_1, \ldots, x_n]$ with usual grading $\deg x_i = 1$.
- The ring $R(\underline{w}) = K[x_1, \ldots, x_n]$ with grading $\deg x_i = w_i$.
- For any element $f \in K[x_1, \ldots, x_n]$ we define $\widetilde{f} \coloneqq f(x_1^{w_1}, \ldots, x_n^{w_n})$.
- The ideal $J = (\partial_1 \widetilde{\mathcal{G}}, \ldots, \partial_n \widetilde{\mathcal{G}}) \subset R$.
- The ideal $I = (\partial_1 \mathcal{G}, \ldots, \partial_n \mathcal{G}) \subset R(\underline{w})$.
- The ideal $\widetilde{I} = (\widetilde{(\partial_1 \mathcal{G})}, \ldots, \widetilde{(\partial_n \mathcal{G})}) \subset R$.

Note that J, I and $\widetilde{I}$ are homogeneous ideals because they are generated by homogeneous elements in their respective rings. Clearly we have

$$J = \left(x_1^{w_1 - 1} \widetilde{(\partial_1 \mathcal{G})}, \ldots, x_n^{w_n - 1} \widetilde{(\partial_n \mathcal{G})} \right).$$

From this we derive the relations

$$J \subset \widetilde{I} \quad \text{and} \quad \left(\prod_{i=1}^{n} x_i^{w_i - 1} \right) \cdot \widetilde{I} \subset J.$$

The next two propositions can be used to compute a basis of the form (4.2.2).

Proposition 4.2.4. *Choose an integer D such that $D - n \geq 0$ as well as a monomial order $\preccurlyeq$ on R. Take $\mathcal{M}_D$ to be the complement of $lm(J_{D-n})$ in the set of all the monomials of degree $D - n$. Then the set $\overline{\mathcal{M}_D}$ consisting of the classes in $\frac{R}{J}$ of the elements of $\mathcal{M}_D$ is a basis for the K-vector space $\left(\frac{R}{J}\right)_{D-n}$.*

Proof. By proposition 4.2.5 below we have

$$R_{D-n} = \mathrm{Span}(\mathcal{M}_D) \oplus \mathrm{Span}(\mathrm{lm}(J_{D-n})) = \mathrm{Span}(\mathcal{M}_D) \oplus \langle \mathrm{lm}(J) \rangle_{D-n}.$$

Now use the fact that the ideals J and $\langle \mathrm{lm}(J) \rangle$ have the same Hilbert function [GP08, Theorem 5.2.6]. It follows that

$$\begin{aligned} |\mathcal{M}_D| &= \dim R_{D-n} - \dim \langle \mathrm{lm}(J) \rangle_{D-n} \\ &= \dim R_{D-n} - \dim J_{D-n} \\ &= \dim \frac{R_{D-n}}{J_{D-n}}. \end{aligned}$$

Also note that no two elements in $\mathcal{M}_D$ can have the same class in $\frac{R_{D-n}}{J_{D-n}}$. Indeed, if m and m' are distinct monomials of degree $D-n$ such that $m-m' \in J_{D-n}$ then either m or m' is the leading monomial of an element of J_{D-n}, hence not in $\mathcal{M}_D$. So we also have $|\overline{\mathcal{M}_D}| = \dim \frac{R_{D-n}}{J_{D-n}}$.

Now we show that the set $\overline{\mathcal{M}_D}$ is linearly independent. For this it suffices to show that for any nontrivial relation

$$\lambda_1 m_1 + \ldots + \lambda_l m_l \in J_{D-n}, \tag{4.2.7}$$

where the m_i are the monomials of degree $D - n$, there exists a coefficient $\lambda_i \neq 0$ such that $m_i \notin \mathcal{M}_D$ or $m_i \in \mathrm{lm}(J_{D-n})$. This is immediate: one of the m_i is the leading monomial of the left-hand side of equation (4.2.7), hence an element of $\mathrm{lm}(J_{D-n})$. □

Proposition 4.2.5. *In R_{D-n} we have an equality of subspaces*

$$\langle \mathit{lm}(J) \rangle_{D-n} = \mathit{Span}(\mathit{lm}(J_{D-n})).$$

Proof. The inclusion "$\supset$" is clear so we only prove the other inclusion.

A general element of $\langle \mathrm{lm}(J) \rangle$ is of the form $\sum_i h_i m_i$ where $m_i = \mathrm{lm}(f_i)$ for some $f_i \in J$ and $h_i \in R$. Since J is a homogeneous ideal we may assume that f_i is homogeneous of the same degree as m_i. An element $\sum_i h_i m_i$ that is homogeneous of degree $D - n$ may be written as $\sum_{i,j} c_{ij} \cdot n_i m_j$ where the n_i are monomials such that $\deg n_i + \deg m_j = D - n$ for every pair (i, j) with $c_{ij} \neq 0$. But this can be rewritten as $\sum_{i,j} c_{ij} \cdot \mathrm{lm}(n_i f_j)$ and since $n_i f_j \in J_{D-n}$, this belongs to $\mathrm{Span}(\mathrm{lm}(J_{D-n}))$. □

Remark 4.2.6. Note that proposition 4.2.4 can be generalized to any homogeneous ideal in a weighted polynomial ring.

Proposition 4.2.4 is in fact very classical. In [Eis95, Theorem 15.3] it is attributed to Macaulay, although the formulation is slightly different. For our applications we will need the exact formulation of propositions 4.2.4 and 4.2.5.

So far we have only used the ideal J, which is the Jacobian ideal "upstairs". The next step is to find a relation with the Jacobian ideal I "downstairs". For this we make the following definition.

Definition 4.2.7. Choose an integer D such that $D - n \geq 0$. Let $\mathrm{Mon}_{\underline{w}}(D)$ denote the set of monomials in $R(\underline{w})$ of weighted degree $D - \sum_i w_i$. Also let $\mathrm{Mon}'(D)$ denote the set of monomials in R of degree $D - n$ that satisfy condition (4.2.1). We define a map

$$\varphi\colon \mathrm{Mon}_{\underline{w}}(D) \to \mathrm{Mon}'(D)\colon x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \mapsto \left(\prod_{i=1}^{n} x_i^{w_i - 1}\right) x_1^{w_1 a_1} \cdot \ldots \cdot x_n^{w_n a_n}.$$

It is easy to see that the map φ is well-defined. Indeed, the fact that a monomial $u = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \in \mathrm{Mon}_{\underline{w}}(D)$ has weighted degree $D - \sum_i w_i$ in $R(\underline{w})$ means that

$$\sum_{i=1}^{n} w_i a_i = D - \sum_{i=1}^{n} w_i. \tag{4.2.8}$$

It follows that $\varphi(u)$ has degree $D - n$. It is clear that $\varphi(u)$ satisfies condition (4.2.1).

It is also easy to show that φ is bijective. Indeed, note that every monomial satisfying (4.2.1) is of the form

$$\left(\prod_{i=1}^{n} x_i^{w_i - 1}\right) x_1^{w_1 a_1} \cdot \ldots \cdot x_n^{w_n a_n} \tag{4.2.9}$$

for some integers $a_1, \ldots, a_n$. The only possible inverse image under φ is

$$x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}. \tag{4.2.10}$$

The fact that the monomial (4.2.9) lies in $\mathrm{Mon}'(D)$ is expressed by the equation (4.2.8). It follows that (4.2.10) has weighted degree $D - \sum_{i=1}^{n} w_i$, as required.

In what follows we will consider the map φ for different values of D. We prefer to keep this dependence implicit in the notation, since the right value for D can usually be deduced from the context.

We can now use the map φ to express a relation between the Jacobi ring R/J "upstairs" and the Jacobi ring $R(\underline{w})/I$ "downstairs".

Proposition 4.2.8. *Choose an integer D such that $D - n \geq 0$. Let $\mathcal{M}_D$ be as in proposition 4.2.4, defined with respect to the lexicographic monomial order $\preccurlyeq_{lex}$. Then apply definition 4.2.1 to obtain a set $\mathcal{M}'_D$. Define $\mathcal{N}_D$ to be the set of monomials of weighted degree $D - \sum_i w_i$ in $R(\underline{w})$ that don't belong to the set $lm(I_{D - \sum_i w_i})$. Then the restriction of φ to the set $\mathcal{N}_D$ defines a bijection*

$$\varphi\colon \mathcal{N}_D \longrightarrow \mathcal{M}'_D.$$

Note that this proposition also holds when $D - n \geq 0$ but $D - \sum_i w_i < 0$. In this case it implies that $\mathcal{M}'_D = \emptyset$.

It is easy to see that the classes of the elements of $\mathcal{N}_D$ form a basis for the K-vector space $(R(\underline{w})/I)_{D-\sum_i w_i}$: the proof of proposition 4.2.4 can be adapted to the weighted homogeneous case. In this sense proposition 4.2.8 can be seen as the algebraic counterpart of the analytical result from proposition 4.1.1. In fact, proposition 4.2.8 can be used to give an algebraic proof of the dimension formula from proposition 4.1.2, removing the need to base-change to $\mathbb{C}$.

We first prove a lemma. This is the only place where we rely on the choice of the lexicographic monomial order $\preccurlyeq_{lex}$. The lemma is certainly false for a general monomial order.

Proposition 4.2.9. *Consider the polynomial ring $K[x_1, \ldots, x_n]$ together with the lexicographic monomial order $\preccurlyeq_{lex}$. Choose an ideal $\mathcal{I} = (f_1, \ldots, f_m)$ and define $\widetilde{\mathcal{I}} = (\widetilde{f}_1, \ldots, \widetilde{f}_m)$. Let $\{Q_1, \ldots, Q_s\}$ denote the unique reduced Gröbner basis of $\mathcal{I}$. Then $\{\widetilde{Q}_1, \ldots, \widetilde{Q}_s\}$ is the reduced Gröbner basis of $\widetilde{\mathcal{I}}$.*

Proof. We can compute a Gröbner basis for $\mathcal{I}$ using Buchberger's algorithm [vzGG13, Algorithm 21.33]. This algorithm boils down to repeatedly computing the remainders of *S-polynomials* w.r.t. a set of generators for $\mathcal{I}$. So it suffices to prove that for any pair (i, j) we have

$$S(\widetilde{f}_i, \widetilde{f}_j) \text{ rem } \widetilde{f}_1, \ldots, \widetilde{f_m} = \widetilde{r_{ij}} \tag{4.2.11}$$

where

$$r_{ij} = S(f_i, f_j) \text{ rem } f_1, \ldots, f_m.$$

Since we are using the lexicographical ordering we have that $\mathrm{lm}(\widetilde{f}) = \widetilde{\mathrm{lm}(f)}$ for any polynomial f. It is then easy to verify that the multivariate division algorithm [vzGG13, Algorithm 21.11] is compatible with the tilde operator. One can also check that

$$S(\widetilde{f}_i, \widetilde{f}_j) = \widetilde{S(f_i, f_j)}.$$

This is a direct consequence of the definition [vzGG13, Definition 21.29]. The equality (4.2.11) follows. To compute the *reduced* Gröbner basis we can use the algorithm described in [vzGG13, Theorem 21.38]. It is easy to see that this algorithm is also compatible with the tilde operator. This finishes the proof. $\square$

Corollary 4.2.10. *Consider the ideal $I = (\partial_1 \mathcal{G}, \ldots, \partial_n \mathcal{G})$ and again choose the lexicographic monomial order $\preccurlyeq_{lex}$. Then the ideal $\langle lm(I) \rangle$ is generated by a set of monomials $\{u_1, \ldots, u_s\}$ such that $\{\widetilde{u}_1, \ldots, \widetilde{u}_s\}$ generates $\langle lm(\widetilde{I}) \rangle$.*

Proof. One defining property of a Gröbner basis $\{Q_1, \ldots, Q_s\}$ of an ideal $\mathcal{I}$ is that $\langle \mathrm{lm}(Q_1), \ldots, \mathrm{lm}(Q_s) \rangle = \langle \mathrm{lm}(\mathcal{I}) \rangle$. See for example [vzGG13, Definition

21.25]. The claim follows immediately from proposition 4.2.9 by taking $u_j \coloneqq \operatorname{lm}(Q_j)$. □

With corollary 4.2.10 we are now ready to give the proof of proposition 4.2.8.

Proof of Proposition 4.2.8. It suffices to work on the complements and to show that φ restricts to a bijection

$$\varphi\colon \operatorname{lm}(I_{D-\sum_i w_i}) \longrightarrow \operatorname{lm}(J_{D-n}) \cap \operatorname{Mon}'(D).$$

By combining corollary 4.2.10 with an argument similar to the proof of proposition 4.2.5 it easily follows that

$$\operatorname{lm}(I_{D-\sum_i w_i}) = \{ \text{ monomials } m \cdot u_j \mid \deg_{\underline{w}} m + \deg_{\underline{w}} u_j = D - \sum_i w_i \}.$$

The elements u_j on the right-hand side are the same as in the corollary. Similarly we have

$$\operatorname{lm}(\widetilde{I}_\beta) = \{ \text{ monomials } m \cdot \widetilde{u}_j \mid \deg m + \deg \widetilde{u}_j = \beta \}$$

for any $\beta \geq 0$. It is now sufficient to show that a monomial $u \in \operatorname{Mon}_{\underline{w}}(D)$ belongs to $\operatorname{lm}(I_{D-\sum_i w_i})$ if and only if $\varphi(u)$ belongs to $\operatorname{lm}(J_{D-n})$.

Assume that $u \in \operatorname{lm}(I_{D-\sum_i w_i})$. Then there exists a monomial m such that $u = m \cdot u_j$ with u_j as in corollary 4.2.10. We then obtain

$$\varphi(u) = \left(\prod_{i=1}^n x_i^{w_i-1}\right) \cdot \widetilde{m} \cdot \widetilde{u}_j \in \left(\prod_{i=1}^n x_i^{w_i-1}\right) \cdot \operatorname{lm}(\widetilde{I}_{D-\sum_i w_i}) \subset \operatorname{lm}(J_{D-n}).$$

Conversely, assume that

$$\varphi(u) = \left(\prod_{i=1}^n x_i^{w_i-1}\right) \cdot x_1^{w_1 a_1} \cdot \ldots \cdot x_n^{w_n a_n} \in \operatorname{lm}(J_{D-n})$$

for some values $a_1, \ldots, a_n$. Since $\operatorname{lm}(J_{D-n}) \subset \operatorname{lm}(\widetilde{I}_{D-n})$ we see that there exists a monomial m such that $\varphi(u) = m \cdot \widetilde{u}_j$ for some u_j as in corollary 4.2.10. Then we must have

$$m = \left(\prod_{i=1}^n x_i^{w_i-1}\right) \cdot \widetilde{m'}$$

for another monomial m'. It follows that $\varphi(u) = \varphi(m' \cdot u_j)$. Hence $u = m' \cdot u_j \in \operatorname{lm}(I_{D-\sum_i w_i})$. □

We can use proposition 4.2.8 together with propositions 4.2.4 and 4.2.2 to efficiently compute a basis for the $G(\underline{w})$-invariant de Rham cohomology space

$$H_{dR}^{n-1}(\mathbb{P}_K^{n-1} \setminus (\widetilde{\mathcal{S}}_K)_\infty)^{G(\underline{w})}. \tag{4.2.12}$$

See algorithm 4.1 below.

Algorithm 4.1: Computing a basis for the $G(\underline{w})$-invariant de Rham cohomology.

Data: A polynomial $\mathcal{G} \in \mathcal{V}[x_1, \ldots, x_n]$ that is weighted homogeneous of degree d w.r.t. weights $\underline{w}$ and that satisfies all the assumptions from the introduction of this chapter.

Result: A set of differential forms whose cohomology classes form a basis for the $G(\underline{w})$-invariant de Rham cohomology (4.2.12).

begin

Compute the reduced Gröbner basis $\{Q_1, \ldots, Q_s\}$ of the ideal

$$I = (\partial_1\mathcal{G}, \ldots, \partial_n\mathcal{G})$$

using the lexicographic monomial order $\preccurlyeq_{lex}$. Then define

$$u_j := \mathrm{lm}(Q_j) \text{ for } 1 \leq j \leq s.$$

for $D \in \{d, 2d, \ldots, (n-1)d\}$ **do**

if $D - \sum_i w_i < 0$ **then**

Define $\mathcal{N}_D := \emptyset$.

else

Define $\mathcal{N}_D$ to be the set of monomials of weighted degree $D - \sum_i w_i$ that are not a multiple of $u_1, \ldots, u_s$.

end

end

Output: the set

$$\bigcup_{\alpha=1}^{n-1} \left\{ \frac{\varphi(u)\,\Omega}{\widetilde{\mathcal{G}}^\alpha} \mid u \in \mathcal{N}_{\alpha d} \right\}.$$

end

The core of algorithm 4.1 is very classical. It goes as far back as the thesis of Buchberger [Buc06]. However, it is the relation to cohomology (see proposition below) that is of interest to us.

Proposition 4.2.11. *The output of algorithm 4.1 is a basis for the invariant de Rham cohomology (4.2.12).*

Proof. The monomials $u_j = \mathrm{lm}(Q_j)$ are precisely those from corollary 4.2.10. Now fix a value of D such that $D - \sum_i w_i \geq 0$. The set $\mathcal{N}_D$ generated by the algorithm is the complement of

$$\{ \text{ monomials } m \cdot u_j \mid \deg_{\underline{w}} m + \deg_{\underline{w}} u_j = D - \sum_i w_i \} \tag{4.2.13}$$

in the set of all monomials of weighted degree $D - \sum_i w_i$. We have shown in the proof of proposition 4.2.8 that the set (4.2.13) is equal to $\operatorname{lm}(I_{D-\sum_i w_i})$. Therefore the set $\mathcal{N}_D$ that is constructed by the algorithm is indeed the one that we defined in the statement of proposition 4.2.8. The result now follows by combining propositions 4.2.8, 4.2.4 and 4.2.2. □

We don't give a detailed analysis of the performance of this algorithm, but it should be clear that it is in general much faster than the naive algorithm that we sketched at the beginning of this paragraph. Indeed, the naive approach comes down to applying the algorithm to $\widetilde{\mathcal{G}}$ instead of $\mathcal{G}$. This produces a larger basis that needs to be trimmed afterwards.

4.2.4 A modified Griffiths-Dwork reduction

In this paragraph we show that the basis from algorithm 4.1 is particularly well suited to carry out the Griffiths-Dwork reduction. Indeed, we will use this basis to rewrite the Griffiths-Dwork reduction algorithm in terms of the Jacobian ideal of the weighted homogeneous equation $\mathcal{G}$. We also show that the coordinates of the cohomology class of a $G(\underline{w})$-invariant form can be directly read off from this modified Griffiths-Dwork method. The key observation is given in the following proposition.

Proposition 4.2.12. *Take a weighted homogeneous polynomial $\mathcal{G}$ as in algorithm 4.1. We use this polynomial to define the ideals*

$$I = (\partial_1 \mathcal{G}, \ldots, \partial_n \mathcal{G}) \quad \text{and} \quad J = (\partial_1 \widetilde{\mathcal{G}}, \ldots, \partial_n \widetilde{\mathcal{G}}).$$

We let $\{Q_1, \ldots, Q_s\}$ denote the reduced Gröbner basis of I w.r.t. $\preccurlyeq_{lex}$. Also write

$$h(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{w_i - 1}.$$

Let $A \in K[x_1, \ldots, x_n]$ be a weighted homogeneous polynomial of weighted degree $td - \sum_i w_i \geq 0$. Now take the remainder

$$r = A \text{ rem } Q_1, \ldots, Q_s$$

using the multivariate division algorithm [vzGG13, Algorithm 21.11] with respect to $\preccurlyeq_{lex}$. Then the following properties hold:

i) The polynomial r is a K-linear combination of the monomials in the set $\mathcal{N}_{td}$ from proposition 4.2.8.

ii) If $t > n-1$ then $r = 0$.

Proof.

i) It is a well-known fact that the reduced Gröbner basis of a weighted homogeneous ideal consists of weighted homogeneous polynomials. One way to prove this property is to analyse Buchberger's algorithm, similarly to the proof of proposition 4.2.9. It follows that the polynomial r is weighted homogeneous as well. But then each monomial in r must have the same weighted degree as A. So r is a K-linear combination of monomials of weighted degree $td - \sum_i w_i$. Also, one of the defining properties of the multivariate remainder is that none of the monomials in r are a multiple of the $u_j = \mathrm{lm}(Q_j)$. It follows that r is indeed a K-linear combination of the monomials in $\mathcal{N}_{td}$.

ii) Assume $t \geq n$, then we have

$$\deg_{\underline{w}} A \geq nd - \sum_i w_i > nd - 2 \cdot \sum_i w_i.$$

It follows that $A \in I$, according to equation (4.1.4). Therefore $r = 0$.

$\square$

We now show how proposition 4.2.12 can be used to implement the Griffiths-Dwork reduction method for a $G(\underline{w})$-invariant differential form that has the shape

$$\frac{h \cdot \widetilde{A}\,\Omega}{\widetilde{\mathcal{G}}^t}.$$

Before we give the algorithm we need to chose an ordering on the monomials in the sets $\mathcal{N}_{\alpha d}$. To this end we write

$$\bigcup_{\alpha=1}^{n-1} \mathcal{N}_{\alpha d} = (v_1, v_2, \dots, v_\delta).$$

With this notation the basis from algorithm 4.1 is given by the differential forms

$$\left\{ \frac{\varphi(v_j)\,\Omega}{\widetilde{\mathcal{G}}^{\alpha_j}} \right\}_j \tag{4.2.14}$$

for $1 \leq j \leq \delta$ and $\alpha_j = \frac{\deg_{\underline{w}} v_j + \sum_i w_i}{d}$. The following algorithm carries out the Griffiths-Dwork reduction with respect to the basis above.

Algorithm 4.2: Griffiths-Dwork reduction with respect to the basis produced by algorithm 4.1.

Data: A weighted homogeneous polynomial $\mathcal{G}$ satisfying the same assumptions as algorithm 4.1 and a weighted homogeneous polynomial $A \in K[x_1, \dots, x_n]$ of weighted degree $td - \sum_i w_i \geq 0$.

Result: A coefficient vector $\underline{\tau} = (\tau_1, \dots, \tau_\delta)$ such that, on the level of cohomology, we have

$$\frac{h \cdot \widetilde{A}\,\Omega}{\widetilde{\mathcal{G}}^t} = \sum_{j=1}^{\delta} \tau_j \, \frac{\varphi(v_j)\,\Omega}{\widetilde{\mathcal{G}}^{\alpha_j}}$$

with the v_j as in equation (4.2.14).

Initialization: Compute the Gröbner basis $\{Q_1, \dots, Q_s\}$ from algorithm 4.1.

Definition of procedure *Reduce(A)* **:**

begin

Compute the remainder $r = A \text{ rem } Q_1, \dots, Q_s$. Then write

$$r = \lambda_1 v_1 + \ldots + \lambda_\delta v_\delta \text{ for } \lambda_i \in K.$$

if $A - r \neq 0$ **then**

Find polynomials $B_1, \dots, B_n$ such that

$$A - r = B_1 \cdot (\partial_1 \mathcal{G}) + \ldots + B_n \cdot (\partial_n \mathcal{G}).$$

Recursively define $\underline{\mu} \coloneqq$ *Reduce* $\left((t-1)^{-1} \cdot \sum_{i=1}^n \partial_i B_i\right)$.

else

Define $\underline{\mu} \coloneqq \underline{0}$.

end

Output: $\underline{\tau} \coloneqq \underline{\lambda} + \underline{\mu}$.

end

We verify that every step in the procedure *Reduce(A)* makes sense. In proposition 4.2.12 we have shown that r can indeed be written as a K-linear combination of the v_j. Moreover, this linear combination can be computed using the multivariate division algorithm [vzGG13, Algorithm 21.11]. The same algorithm allows us to write $A - r$ as a combination of the Q_j. During the computation of the Gröbner basis one can keep track of how the Q_j can be written as combination of the $\partial_i \mathcal{G}$, and this gives the polynomials B_i. It is easy to see that the B_i are again weighted homogeneous. Their degree is equal to

$$\deg_{\underline{w}} B_i = td - \sum_i w_i - (d - w_i)$$

and therefore

$$\deg_{\underline{w}} \partial_i B_i = \deg_{\underline{w}} B_i - w_i = (t-1)d - \sum_i w_i.$$

This shows that the recursive call to *Reduce* makes sense. Now we are ready to show that the output of algorithm 4.2 is correct.

Proposition 4.2.13. *Consider an input A of weighted degree $td - \sum_i w_i \geq 0$ for algorithm 4.2, which generates an output $\underline{\tau} = (\tau_1, \ldots, \tau_\delta)$. Then on the level of cohomology we have a linear combination*

$$\frac{h \cdot \widetilde{A}\,\Omega}{\widetilde{\mathcal{G}}^t} = \sum_{j=1}^{\delta} \tau_j \, \frac{\varphi(v_j)\,\Omega}{\widetilde{\mathcal{G}}^{\alpha_j}}.$$

Proof. Using the notations of the algorithm, we write

$$\begin{aligned} h\,\widetilde{A} &= h\,\widetilde{B}_1 \cdot \widetilde{(\partial_1 \mathcal{G})} + \ldots + h\,\widetilde{B}_n \cdot \widetilde{(\partial_n \mathcal{G})} + h\,\widetilde{r} \\ &= \frac{h_1}{w_1}\,\widetilde{B}_1 \cdot \partial_1 \widetilde{\mathcal{G}} + \ldots + \frac{h_n}{w_n}\,\widetilde{B}_n \cdot \partial_n \widetilde{\mathcal{G}} + h\,\widetilde{r} \end{aligned}$$

with

$$h_i = x_1^{w_1 - 1} \cdot \ldots \cdot \widehat{x_i^{w_i - 1}} \cdot \ldots \cdot x_n^{w_n - 1}.$$

By combining propositions 4.2.12 and 4.2.8 we see that the differential form

$$\frac{h \cdot \widetilde{r}\,\Omega}{\widetilde{\mathcal{G}}^t}$$

is a linear combination of the differential forms from the set (4.2.14). The coefficients of this linear combination are the same λ_j as in the algorithm. It remains to verify that the recursive part of the algorithm correctly applies the pole order reduction rule from [Gri69] to the differential form

$$\frac{h \cdot (\widetilde{A} - \widetilde{r})\,\Omega}{\widetilde{\mathcal{G}}^t}.$$

By linearity it suffices to check this for each term individually. For $1 \leq i \leq n$ the pole order reduction rule reads

$$\frac{\frac{h_i}{w_i}\widetilde{B}_i \cdot (\partial_i \widetilde{\mathcal{G}})\,\Omega}{\widetilde{\mathcal{G}}^t} \equiv (t-1)^{-1}\,\frac{\partial_i \left(\frac{h_i}{w_i}\widetilde{B}_i\right)\Omega}{\widetilde{\mathcal{G}}^{t-1}}.$$

But we have

$$\begin{aligned} \partial_i (h_i \widetilde{B}_i) &= \partial_i h_i \cdot \widetilde{B}_i \quad + \quad h_i \cdot \partial_i \widetilde{B}_i \\ &= 0 \qquad\qquad + \quad h_i \cdot w_i\, x_i^{w_i - 1} \widetilde{(\partial_i B_i)}. \end{aligned}$$

Therefore we find that

$$\partial_i \left(\frac{h_i}{w_i} \widetilde{B}_i \right) = h \cdot \widetilde{(\partial_i B_i)}.$$

From this we see that the recursive call *Reduce* $\left((t-1)^{-1} \cdot \sum_{i=1}^{n} \partial_i B_i\right)$ indeed corresponds to the classical pole order reduction rule. $\square$

Remark 4.2.14. The remainders r that are computed by algorithm 4.2 can be seen as an analogue of the Griffiths-Dwork reduction for weighted homogeneous polynomials of degree $td - \sum_i w_i$. In this sense algorithm 4.2 is a generalization of [BLS13, Algorithm 1], which computes the Griffiths-Dwork reduction of the differential form associated to a *homogeneous* polynomial. It follows from proposition 4.2.13 that two weighted homogeneous polynomials A and A' define the same cohomology class in $H_{dR}^{n-1}(\mathbb{P}_K^{n-1} \setminus (\widetilde{\mathcal{S}}_K)_\infty)^{G(\underline{w})}$ if and only if their reduced forms are equal. This is analogous to [BLS13, Theorem 1].

Remark 4.2.15. Algorithm 4.2 can easily be applied to the truncated sum (4.2.6). Indeed, it is easy to calculate that a differential form of the shape

$$\frac{x_1^{q-1} \cdot \ldots \cdot x_n^{q-1} \cdot F(\varphi(v_j)) \cdot (\widetilde{\mathcal{G}}^q - F(\widetilde{\mathcal{G}}))^i \, \Omega}{\widetilde{\mathcal{G}}^{q(i+\alpha)}}$$

can be rewritten as

$$\frac{h \cdot \widetilde{A} \, \Omega}{\widetilde{\mathcal{G}}^{q(i+\alpha)}}$$

with

$$A = x_1^{q-1} \cdot \ldots \cdot x_n^{q-1} \cdot F(v_j) \cdot (\mathcal{G}^q - F(\mathcal{G}))^i.$$

With algorithms 4.1 and 4.2 and the remark above we now have enough details to implement the modified AKR algorithm. Our implementation in SAGE can be found at the address

`https://github.com/ouwehand`

Note that this implementation is only meant to demonstrate the algorithms from this chapter. All the other details of the AKR algorithm are kept as simple as possible.

We have limited the implementation to polynomials $\mathcal{G} \in \mathbb{Z}[x_1, \ldots, x_n] \subset \mathbb{Z}_p[x_1, \ldots, x_n]$, which reduce to $g \in \mathbb{F}_p[x_1, \ldots, x_n]$. The bottleneck of our implementation is that all the intermediate computations are over $\mathbb{Q}$. Only the final result is interpreted as a matrix with entries in $K = \mathbb{Q}_p$. It is more efficient to use approximate p-adic arithmetic, but this also causes some additional loss of precision. This issue is not discussed in [AKR11], but the original implementation[3] contains some comments about it.

[3]Available from the homepage of Kiran Kedlaya.

Our implementation can be used on simple yet interesting examples, despite the limitations on speed and generality.

Example 4.2.16. Consider for instance the following equation in three variables over the base field $k = \mathbb{F}_3$:

$$g = x_1^5 + x_2^{10} + x_3^2 + x_1 x_2^3 x_3.$$

This equation belongs to the weighted Dwork family, whose rigid cohomology is studied in the paper [Klo07]. The weights are given by $\underline{w} = (2, 1, 5)$ and the weighted degree d is equal to 10. The modified AKR algorithm can be used to compute an approximation of the characteristic polynomial of Frobenius, based on the first 10 terms of the sum (4.2.6). This computation finishes quickly, even on old hardware. The resulting approximation suggests that the characteristic polynomial of the Frobenius action on $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}$ should be:

$$T^4 - 3\,T^3 - 3^4\,T + 3^6.$$

4.2.5 Changing the monomial order

Most of the proofs in paragraphs 4.2.3 and 4.2.4 strongly depend on the assumption that one uses the lexicographical monomial order $\preccurlyeq_{lex}$. A natural question to ask is: what happens if we choose a different monomial order?

Say we work with the degree lexicographical order $\preccurlyeq_{deglex}$. It is immediately obvious that the tilde operator is no longer compatible with the monomial order. In fact, it is not difficult to give an example for which the statements of propositions 4.2.9, 4.2.10 and 4.2.8 are all false. More precisely, if we define the sets $\mathcal{N}_D$ and $\mathcal{M}'_D$ with respect to the order $\preccurlyeq_{deglex}$, then $\varphi(\mathcal{N}_D)$ is generally not equal to $\mathcal{M}'_D$. In case of example 4.2.16, we have $\varphi(\mathcal{N}_D) \not\subset \mathcal{M}'_D$ and $\mathcal{M}'_D \not\subset \varphi(\mathcal{N}_D)$ at the same time.

On the other hand, it is quite easy to see that algorithm 4.1 still produces a basis for the $G(\underline{w})$-invariant de Rham cohomology. Indeed, this follows immediately from the following proposition.

Proposition 4.2.17. *Take a polynomial $\mathcal{G} \in \mathcal{V}[x_1, \ldots, x_n]$ that is weighted homogeneous of degree d w.r.t. weights $\underline{w}$ and that satisfies all the usual assumptions. For $D \in \{d, 2d, \ldots, (n-1)d\}$, choose a set $\mathcal{B}_D$ of monomials in $R(\underline{w})$ whose classes form a basis for the K-vector space*

$$\left(\frac{R(\underline{w})}{I}\right)_{D - \sum_i w_i}.$$

Then the set

$$\bigcup_{\alpha=1}^{n-1} \left\{ \frac{\varphi(m)\,\Omega}{\widetilde{\mathcal{G}}^\alpha} \mid m \in \mathcal{B}_{\alpha d} \right\} \tag{4.2.15}$$

forms a basis for the $G(\underline{w})$-invariant de Rham cohomology

$$H^{n-1}_{dR}(\mathbb{P}^{n-1}_K \setminus (\widetilde{\mathcal{S}}_K)_\infty)^{G(\underline{w})}.$$

Proof. First observe that the set (4.2.15) has the right number of elements, according to proposition 4.1.1. We only need to verify that the set (4.2.15) is linearly independent.

So assume that there is a nontrivial linear combination on the level of cohomology:

$$\sum_i \lambda_i \frac{\varphi(m_i)\,\Omega}{\widetilde{\mathcal{G}}^{\alpha_i}} = \frac{\sum_i \lambda_i\,\varphi(m_i)\,\widetilde{\mathcal{G}}^{t-\alpha_i}\,\Omega}{\widetilde{\mathcal{G}}^t} \equiv 0. \qquad (4.2.16)$$

Here we have chosen an arbitrary ordering of the elements in $\cup_\alpha \mathcal{B}_{\alpha d}$, with $m_i \in \mathcal{B}_{\alpha_i d}$. The value t is defined by the formula:

$$t = \max\{\alpha_i \mid \lambda_i \neq 0\}.$$

This quantity is well-defined, since we assumed that at least one of the λ_i is nonzero. We may then write the numerator on the left-hand side of equation (4.2.16) as $h \cdot \widetilde{A}$, where

$$A = \sum_i \lambda_i\, m_i\, \mathcal{G}^{t-\alpha_i}.$$

Now we consider the modified Griffiths-Dwork reduction from remark 4.2.14. Since A induces the zero class in cohomology, its reduction must be zero. In particular, we have $A \in I$.

By collecting the indices i for which $\alpha_i = t$, we may rewrite this as

$$\sum_j \lambda_j\, m_j + \sum_l \lambda_l\, m_l\, \mathcal{G}^{\beta_l} \in I$$

with $\beta_l > 0$ for all l. Since also $\mathcal{G} \in I$, we find a nontrivial linear combination

$$\sum_j \lambda_j\, m_j \in I.$$

But this is in contradiction with the fact that these monomials m_j belong to $\mathcal{B}_{td}$. This concludes the proof. □

The proof above should be compared with [BLS13, Proposition 2]. The only real difference is that we replaced the classical Griffiths-Dwork reduction with the modified version from remark 4.2.14. As a result, the proof of proposition 4.2.17 heavily depends on the theory for $\preccurlyeq_{lex}$ that we discussed in paragraphs 4.2.3 and 4.2.4.

Proposition 4.2.17 clearly implies that algorithm 4.1 produces a basis for any choice of monomial order $\preccurlyeq$, since the sets $\mathcal{N}_D$ (defined w.r.t. $\preccurlyeq$) form a basis for $(R(\underline{w})/I)_{D-\sum_i w_i}$. This last claim can be proved similarly as proposition 4.2.4.

This basis is still special, in the sense that algorithm 4.2 correctly computes the coordinates w.r.t. this basis. To see this, consider the sets $\mathcal{N}_D$, defined w.r.t. a general order $\preccurlyeq$. Then the proof of point i) of proposition 4.2.12 goes through without any modification. The statement of point ii) is clearly independent of the choice of monomial order, since $r = 0$ if and only if $A \in I$. As a result, the proof of proposition 4.2.13 is still valid for the basis that is computed by algorithm 4.1 w.r.t. a general order $\preccurlyeq$.

These observations form the basis for the following proposition:

Proposition 4.2.18. *The modified AKR algorithm is still correct if one replaces $\preccurlyeq_{lex}$ with any global monomial order $\preccurlyeq$.*

Proof. We have already shown that

$$\bigcup_{\alpha=1}^{n-1} \left\{ \frac{\varphi(m)\,\Omega}{\widetilde{\mathcal{G}}^{\alpha}} \mid m \in \mathcal{N}_{\alpha d} \right\} \tag{4.2.17}$$

is a basis for $H_{dR}^{n-1}(\mathbb{P}_K^{n-1} \setminus (\widetilde{\mathcal{S}}_K)_\infty)^{G(\underline{w})}$ and that algorithm 4.2 correctly computes the coordinates w.r.t. this basis. It remains to show that the sets $\varphi(\mathcal{N}_{\alpha d})$ can be obtained by applying definition 4.2.1 to a set of monomials $\mathcal{M}_{\alpha d}$ whose classes form a basis for $(R/J)_{\alpha d - n}$. The results in paragraph 4.2.1 then guarantee that the precision estimates from [AKR11] still hold.

It is easy to see that the set $\varphi(\mathcal{N}_{\alpha d})$ is linearly independent in $(R/J)_{\alpha d-n}$. Indeed, for a fixed value α, consider a linear combination

$$\lambda_1\,\varphi(v_1) + \ldots + \lambda_l\,\varphi(v_l) \in J$$

for $v_1, \ldots, v_l \in \mathcal{N}_{\alpha d}$. This implies that the differential form

$$\frac{\sum_{i=1}^{l} \lambda_i\,\varphi(v_i)\,\Omega}{\widetilde{\mathcal{G}}^{\alpha}}$$

is cohomologous to a form of pole order $\alpha - 1$. But this is in contradiction with the basis (4.2.17), unless $\lambda_1 = \ldots = \lambda_l = 0$.

Now fix an $1 \le \alpha \le n-1$ and consider the subspace $V_{\alpha d-n} \subset R_{\alpha d - n}$ that is spanned by the monomials of degree $\alpha d - n$ that don't belong to $\varphi(\mathcal{N}_{\alpha d})$. We then obtain a decomposition

$$\left(\frac{R}{J}\right)_{\alpha d - n} = \mathrm{Span}(\varphi(\mathcal{N}_{\alpha d})) \oplus \frac{V_{\alpha d-n}}{J_{\alpha d-n}}.$$

Choose a set $\mathcal{C}_{\alpha d}$ of monomials in $V_{\alpha d-n}$ whose classes form a basis for $\frac{V_{\alpha d-n}}{J_{\alpha d-n}}$. Then the set

$$\mathcal{M}_{\alpha d} = \varphi(\mathcal{N}_{\alpha d}) \cup \mathcal{C}_{\alpha d}$$

is a set of monomials whose classes form a basis for $(R/J)_{\alpha d-n}$. After applying definition 4.2.1 we obtain an obvious inclusion $\varphi(\mathcal{N}_{\alpha d}) \subset \mathcal{M}'_{\alpha d}$. By combining proposition 4.2.2 and the fact that (4.2.17) is a basis, we conclude that in fact $\varphi(\mathcal{N}_{\alpha d}) = \mathcal{M}'_{\alpha d}$. This finishes the proof. $\square$

We emphasize again that the proof of proposition 4.2.18 strongly relies on the modified Griffiths-Dwork reduction from remark 4.2.14, which is based on the lexicographic monomial order. The arguments used in this paragraph, considered on their own, only prove the following *a priori* weaker statement:

Proposition 4.2.19. *If the modified AKR algorithm works for* some *monomial order, then it works for* any *monomial order.*

Our implementation of the modified AKR algorithm includes a demonstration of the phenomena that occur when the monomial order is changed. One surprising observation is that the choice of the monomial order can affect the precision of the approximate Frobenius matrix. For example 4.2.16, the monomial orders $\preccurlyeq_{deglex}$ and $\preccurlyeq_{degrevlex}$ both give more precision than $\preccurlyeq_{lex}$ for the same level of truncation in the sum (4.2.6). Also, the basis for the $G(\underline{w})$-invariant de Rham cohomology that is computed with respect to $\preccurlyeq_{degrevlex}$ happens to be the same as the basis w.r.t. $\preccurlyeq_{lex}$. This shows that the monomial order is really responsible for the gained precision.

Thus it seems that $\preccurlyeq_{degrevlex}$ is a good overall choice of monomial order. It is generally considered a good choice for computing Gröbner bases, and it seems to give better results than $\preccurlyeq_{lex}$ when combined with the modified AKR algorithm.

Remark 4.2.20. We should mention that there is an alternative proof for proposition 4.2.18, which works by modifying propositions 4.2.9, 4.2.10 and 4.2.8. To do this, one should use a different monomial order “upstairs” that is adapted to the choice of monomial order “downstairs”. If one represents the monomial order “downstairs” by an element of $\mathrm{GL}(n, \mathbb{R})$, then it is easy to see which matrix should represent the monomial order “upstairs”.

Our approach was to use the modified Griffiths-Dwork reduction from remark 4.2.14, after which it suffices to work “downstairs”. The advantage of this approach is that it allows us to use the classical proofs from [BLS13] with almost no modifications.

4.2.6 Comparison with the *Frobenius project*

The Frobenius project [dJ06] is a program written in C whose stated goal is to approximate the Frobenius action on the rigid cohomology

$$H^{n-1}_{rig}(\mathbb{P}_k(\underline{w}) \setminus S_\infty),$$

with S_∞ a quasi-smooth weighted projective hypersurface w.r.t. weights $\underline{w} = (w_1, \ldots, w_n)$ over a finite field k.

The code from [dJ06] comes with a text file that briefly explains what is being computed. To our knowledge there are no rigorous proofs for the correctness of this code. For this reason we start by proving that the algorithm behind [dJ06] correctly calculates a reduction of certain formal differentials.

Reduction of formal differentials

Consider a weighted homogeneous polynomial $\mathcal{G} \in K[x_1, \ldots, x_n]$ defining a quasi-smooth weighted projective hypersurface $(\widetilde{\mathcal{S}}_K)_\infty \subset \mathbb{P}_K(\underline{w})$ of degree d. Then define the K-vector space of *formal* differentials

$$Z^{n-1} = \left\{ \frac{A\,\Omega}{\mathcal{G}^t} \mid \deg_{\underline{w}} A = td - \sum_{i=1}^n w_i \right\}$$

where

$$\Omega = \sum_{i=1}^n (-1)^{i-1}\, w_i\, x_i \cdot \Big(dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \Big).$$

The *formal cohomology* H^{n-1}_{formal} is defined as the quotient of Z^{n-1} by the subspace of differentials that arise as a (formal) derivative. More precisely: $H^{n-1}_{\text{formal}} = Z^{n-1}/B^{n-1}$ where B^{n-1} is the subspace spanned by elements of the form

$$\frac{A \cdot (\partial_i \mathcal{G})\,\Omega}{\mathcal{G}^t} - (t-1)^{-1}\, \frac{\partial_i A\,\Omega}{\mathcal{G}^{t-1}},$$

for $1 \leq i \leq n$ and $t \geq 2$.

The assumption that $\mathcal{G}$ is quasi-smooth ensures that the results from paragraph 4 of [Gri69] still hold. To see this it suffices to verify the following two properties:

i) If $\deg_{\underline{w}} A \geq nd - \sum_{i=1}^n w_i$, then $A \in I = (\partial_1 \mathcal{G}, \ldots, \partial_n \mathcal{G})$.

ii) If $\sum_{i=1}^n B_i \cdot \partial_i \mathcal{G} = 0$ for certain B_i, then there exists a skew-symmetric matrix of polynomials C_{ij} satisfying the property

$$B_i = \sum_{j=1}^n C_{ij} \cdot \partial_j \mathcal{G}$$

for every $1 \leq i \leq n$.

These properties can be used as substitutes for [Gri69, Theorem 4.11] and [Gri69, Proposition 4.14] (originally due to Macaulay and Dwork). The other proofs in paragraph 4 of [Gri69] then go through without any modifications.

We have already proved the first of the two claims above, see equation (4.1.4). For this property to hold one only needs the fact that the partial derivatives $\partial_1 \mathcal{G}, \ldots, \partial_n \mathcal{G}$ form a regular sequence. The second claim is already covered by [Dwo62, Lemma 3.1], the same lemma that is cited in [Gri69]. Again, this statement is valid for any regular sequence.

By repeating the proofs from [Gri69], one can now show that the differentials in Z^{n-1} admit a Griffiths-Dwork reduction. An element of Z^{n-1} induces the zero class in H^{n-1}_{formal} if and only if its reduction is zero.

The existence of such a Griffiths-Dwork reduction implies that H^{n-1}_{formal} admits a basis that is similar to the one from proposition 1.2.4. The proof is formally the same as in [BLS13, Proposition 2]. It is then not difficult to devise an algorithm that writes an element of Z^{n-1} as a linear combination of this basis, similarly to the homogeneous case. This (among other things) is implemented in [dJ06].

Relation with rigid cohomology

The statement that the code from [dJ06] can be used to approximate the Frobenius action on the rigid cohomology $H^{n-1}_{rig}(\mathbb{P}_k(\underline{w}) \setminus S_\infty)$ is much less obvious.

Over the complex numbers it is known that there is an identification

$$H^{n-1}_{\text{formal}} \otimes \mathbb{C} \xrightarrow{\sim} H^{n-1}(\mathbb{P}_{\mathbb{C}}(\underline{w}) \setminus (\mathcal{S}_{\mathbb{C}})_\infty, \mathbb{C}).$$

This correspondence can be proved using the techniques from the proof of proposition 3.4.1.

But in remark 3.4.2 we have explained that (outside of the homogeneous case) we see no clear connection between the rigid cohomology of weighted projective hypersurfaces and formal differentials. For this reason we believe that additional theoretical foundations are needed to prove that the algorithm behind [dJ06] correctly approximates the Frobenius on $H^{n-1}_{rig}(\mathbb{P}_k(\underline{w}) \setminus S_\infty)$.

On the other hand we found that under additional assumptions the code from the Frobenius project is computationally equivalent to our implementation of the modified AKR algorithm. More precisely: assume that the hypersurface $S_\infty = Z_{\mathbb{P}_k(\underline{w})}(g)$ satisfies all the assumptions from definition 3.1.7, in particular that $\widetilde{S}_\infty$ is smooth. Then the code from [dJ06] computes the same approximate matrix as our algorithms 4.1 and 4.2 applied to the truncated Frobenius lift (4.2.6).

To see this, let $\mathcal{G}$ be a weighted homogeneous lift of the equation g. Also consider a set of monomials $v_1, \dots, v_\delta$ whose classes form a basis for H^{n-1}_{formal}. The final output of the Frobenius project is the matrix that is obtained by writing the formal differentials

$$q^{n-1} \sum_{i=0}^{N} \binom{\alpha_j + i - 1}{i} \frac{x_1^{q-1} \cdot \ldots \cdot x_n^{q-1} \cdot F(v_j) \cdot (\mathcal{G}^q - F(\mathcal{G}))^i \, \Omega}{\mathcal{G}^{q(i+\alpha_j)}} \tag{4.2.18}$$

as a linear combination of the basis $v_1, \ldots, v_\delta$. The value N at which the sum is truncated is a parameter of the program.

The first observation is that the numerators of the formal differentials (4.2.18) are the same weighted homogeneous polynomials that we encountered in remark 4.2.15. So the truncated sums (4.2.6) can essentially be obtained by "pulling up" the formal differentials (4.2.18).

The elements v_j are computed by the function `char_0_basis`. At least, this is the case if the degree $d = \deg_{\underline{w}} \mathcal{G}$ is not divisible by the characteristic of k. One can verify that this function is computationally equivalent to our algorithm 4.1. So the elements v_j are given by the same sets of monomials $\mathcal{N}_D$ as computed by algorithm 4.1.

The pole order reduction rule used in [dJ06] reads

$$\frac{B(\partial_i \mathcal{G})\,\Omega}{\mathcal{G}^t} \equiv (t-1)^{-1} \frac{(\partial_i B)\,\Omega}{\mathcal{G}^{t-1}}. \tag{4.2.19}$$

This is essentially the same rule as is applied by our algorithm 4.2. It should be noted that the Frobenius project uses a different definition of the form Ω, which ensures that the pole order reduction rule (4.2.19) works "downstairs", instead of "upstairs" as in proposition 4.2.13. But this different definition of Ω doesn't affect the matrix that is being computed.

So when $\widetilde{S}_\infty$ is smooth, we see that applying the pole order reduction (4.2.19) to the forms (4.2.18) amounts to the same thing as applying the classical pole order reduction to the truncated forms (4.2.6). In other words, there is an identification

$$H^{n-1}_{\text{formal}} \xrightarrow{\sim} H^{n-1}_{dR}(\mathbb{P}^{n-1}_K \setminus (\widetilde{\mathcal{S}}_K)_\infty)^{G(\underline{w})}$$

and the algorithm from [dJ06] corresponds to the modified AKR algorithm. We conclude that under the assumptions from definition 3.1.7, both algorithms compute the same matrix. Possibly up to a permutation of the rows and columns, due to the fact that the basis elements can be permuted[4].

This suggests that in the case where $\widetilde{S}_\infty$ is smooth, [dJ06] implicitly relies on the assumption that there is a Frobenius-equivariant isomorphism of rigid cohomology

$$H^{n-1}_{rig}(\mathbb{P}_k(\underline{w}) \setminus S_\infty) \xrightarrow{\sim} H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}. \tag{4.2.20}$$

We have already noted in remark 3.4.2 that such an identification is far from obvious outside of the homogeneous case where $\underline{w} = (1, \ldots, 1)$. If S_∞ is quasi-smooth but $\widetilde{S}_\infty$ is not smooth then the justification would need an analogue in rigid cohomology of the decomposition (3.4.2) in terms of modified differen-

[4]Also note that the implementation from [dJ06] multiplies the Frobenius matrix with an extra factor q^{-1}.

tials. This seems even more difficult to prove than the isomorphism (4.2.20).

Aside from this issue, it has been commented in [Ked12] that the Frobenius project doesn't provide precision estimates. We have shown at the end of paragraph 4.2.1 that the original estimates from [AKR11] can be used if one assumes that $\widetilde{S}_\infty$ is smooth, although these bounds can probably be significantly improved.

4.2.7 The characteristic polynomial of Frobenius

The algorithms in paragraphs 4.2.3 and 4.2.4 allow us to introduce another computable invariant of a weighted homogeneous hypersurface singularity: the *approximate characteristic polynomial of Frobenius.* We start by giving the necessary definitions. After that we give an algorithm to compute the approximate characteristic polynomial. This is quite straightforward, but one needs to be aware of the possibility that some p-adic precision is lost.

The approximate characteristic polynomial

Definition 4.2.21. Given a weighted homogeneous hypersurface singularity $Y = Z_{\mathbb{A}^n_k}(g)$, we define the characteristic polynomial of Frobenius as

$$P(T) = \det\left(T \cdot \mathrm{Id} - \mathrm{Fr} \mid H^n_{rig,\{0\}}(Y)\right) \in K[T].$$

Unlike the matrix of Frobenius, which depends on a choice of basis, the polynomial $P(T)$ only depends on the Frobenius action on $H^n_{rig,\{0\}}(Y)$. Also, it follows from theorem 2.1.1 that this polynomial is invariant under contact equivalence. Therefore the characteristic polynomial of Frobenius is indeed an invariant of the singularity.

It is obvious that $P(T)$ is related to the Frobenius action on the $G(\underline{w})$-invariant part of $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$.

Proposition 4.2.22. *Let $P(T)$ denote the characteristic polynomial of Frobenius of a weighted homogeneous singularity $Y = Z_{\mathbb{A}^n_k}(g)$. As usual we assume that all the conditions of definition 3.1.7 hold. If $n \geq 3$ then*

$$P(T) = \det\left(T \cdot Id - Fr \mid H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}\right).$$

For $n = 2$ we have

$$P(T) = (T - q) \cdot \det\left(T \cdot Id - Fr \mid H^1_{rig}(\mathbb{P}^1_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}\right).$$

Proof. For $n \geq 3$ this is an immediate consequence of theorem 3.1.11. The case $n = 2$ follows from remark 3.5.11. □

With this description of the characteristic polynomial of Frobenius we can show that $P(T)$ has coefficients in $\mathcal{V} \subset K$. Note that this property is not immediate, because the matrix of the Frobenius action on $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$ can have entries that do not belong to $\mathcal{V}$.

Proposition 4.2.23. *The the characteristic polynomial $P(T)$ from definition 4.2.21 belongs to $\mathcal{V}[T]$.*

Proof. First we show that the polynomial

$$\det\left(\mathrm{Id} - T \cdot \mathrm{Fr} \mid H^{n-2}_{rig}(\widetilde{S}_\infty)\right) \tag{4.2.21}$$

belongs to $\mathbb{Z}[T]$. Since $\widetilde{S}_\infty$ is proper this polynomial can be seen as a part of the zeta function of $\widetilde{S}_\infty$. More precisely, this follows from [ÉLS93, Théorème 6.3], which relates zeta functions to rigid cohomology with compact supports[5]. From the p-adic interpretation of the Weil theorem (see paragraph 6.6 of [Ked06]) one deduces that there cannot be any cancellation in the zeta function of $\widetilde{S}_\infty$, so that (4.2.21) must belong to $\mathbb{Z}[T]$. The details of the argument are the same as in the first paragraph of [Del74].

Now consider the polynomial

$$Q(T) = \det\left(\mathrm{Id} - T \cdot \mathrm{Fr} \mid H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)\right).$$

From the Frobenius-equivariant short exact sequence

$$0 \longrightarrow H^{n-1}_{rig}(\mathbb{P}^{n-1} \setminus \widetilde{S}_\infty) \longrightarrow H^{n-2}_{rig}(\widetilde{S}_\infty)(-1) \longrightarrow H^{n}_{rig}(\mathbb{P}^{n-1}) \longrightarrow 0$$

we can deduce that

$$\det\left(\mathrm{Id} - q\,T \cdot \mathrm{Fr} \mid H^{n-2}_{rig}(\widetilde{S}_\infty)\right) = Q(T) \cdot (1 - q^{\frac{n}{2}} T)^{\delta(n)}$$

where

$$\delta(n) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

It follows that $Q(T) \in \mathbb{Z}[T]$. We now write

$$R(T) = \det\left(T \cdot \mathrm{Id} - \mathrm{Fr} \mid H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)\right).$$

Since $R(T) = T^{\deg Q} \cdot Q\left(\frac{1}{T}\right)$ we have $R(T) \in \mathbb{Z}[T]$. We may assume that $n \geq 3$, so that

$$P(T) = \det\left(T \cdot \mathrm{Id} - \mathrm{Fr} \mid H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}\right).$$

[5] The formula (1.2.2) from the introduction is a special case of the second statement of [ÉLS93, Théorème 6.3], which relates the zeta function of a *smooth* scheme to the rigid cohomology without supports. For smooth proper schemes one finds two different ways to write the zeta function, which gives a proof of the *functional equation*.

We also know that $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}$ is a Frobenius-stable K-subspace of $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)$. It immediately follows that $P(T) \mid R(T)$ in the ring $K[T]$. The fact that $P(T) \in \mathcal{V}[T]$ now follows from a generalization of Gauß' lemma, see [MRR88, corollary IV.4.4]. □

Before we proceed we will need to recall some facts about approximate p-adic arithmetic. These can also be found in the second paragraph of [Vac14].

Definition 4.2.24. Fix an element $z \in K$. We say that $z_0 \in K$ is an approximation of z with *absolute precision* $r \geq 1$ if $z - z_0 \in p^r\mathcal{V}$. The integer r should be thought of as a required level of precision that has been fixed in advance. In other words: there may be a larger r' such that $z - z_0 \in p^{r'}\mathcal{V}$. Also, it is not really necessary to name the approximate element z_0. One can say that z *is known with absolute precision* r.

If an element $z \in \mathcal{V}$ is known with absolute precision r, then this element defines a unique element of the ring $\mathcal{V}/(p^r)$. Indeed, the element $(z_0 \bmod p^r) \in \mathcal{V}/(p^r)$ is the same for any approximation $z_0 \in \mathcal{V}$ satisfying $z - z_0 \in p^r\mathcal{V}$.

If $z_1 \in K$ and $z_2 \in K$ are known with absolute precision r, then so is $z_1 + z_2$. If $z_1 \in \mathcal{V}$ and $z_2 \in \mathcal{V}$ are known with absolute precision r, then so is $z_1 \cdot z_2$. This means that for $z_1, z_2 \in \mathcal{V}$ with precision r, we may regard addition and multiplication as operations in the ring $\mathcal{V}/(p^r)$.

Thus proposition 4.2.23 ensures that an approximation of precision m of the characteristic polynomial $P(T)$ can be considered as an element of $\mathcal{V}[T]/(p^m)$. This leads to the following definition.

Definition 4.2.25. Given an integer $m \geq 1$, we define the *approximate characteristic polynomial of Frobenius*

$$P_m(T) \coloneqq P(T) \bmod p^m,$$

which is an element of the ring $\mathcal{V}[T]/(p^m)$.

For a fixed choice of m, the polynomial $P_m(T)$ is an invariant of the weighted homogeneous singularity $(Y, 0)$. It is quite obvious that the polynomials $P_m(T)$ can be computed using the modified AKR algorithm. We only need to determine the necessary precision of the Frobenius matrix, as a function of m.

Bounds on precision loss

It is well-known that the matrix that is computed by the (modified) AKR algorithm may have denominators. These denominators will cause some loss of precision when we compute the (approximate) characteristic polynomial. On the other hand, it is known that the denominators in the Frobenius matrix are bounded. This allows us to determine bounds for the precision that is lost, see algorithm 4.3 below. Our approach is very straightforward, but we did not find a good reference for it.

Remark 4.2.26. The ideal situation would be to have a basis for which the Frobenius matrix has entries in $\mathcal{V}$. Then there would be no loss of precision when we compute the characteristic polynomial. It is explained in [AKR11] that there is such a basis, coming from crystalline cohomology. The best known description of this basis is presented in [Min13, Theorem B]. A special case of this result has been used in [vdB08] to study hyperelliptic curves. From this one obtains an algorithm to effectively calculate an integral basis for such a curve, see the discussion just after [vdB08, Proposition 5.4]. It seems that the general situation, as described in [Min13, Theorem B], is still too abstract to be turned into a general-purpose algorithm. However, this result does have a number of immediate applications. See [Min13, Remark 3.8], and in particular the application to superelliptic curves that is given in [Min13, Proposition 4.2].

Remark 4.2.27. We should point out that there are *two* ways in which precision loss can occur when we calculate the characteristic polynomial of Frobenius. Indeed, the q-power Frobenius matrix is usually computed as a product

$$M \cdot M^{\sigma} \cdot M^{\sigma^2} \cdot \ldots \cdot M^{\sigma^{v_p(q)-1}} \tag{4.2.22}$$

where M is the matrix of the p-power Frobenius. The precision of M can be controlled using the algorithm at the beginning of paragraph 3.5 in [AKR11]. Since the matrix M can have denominators, there may be some loss of precision in the calculation of (4.2.22). This loss of precision can be bounded. For hyperelliptic curves this is discussed in [Ked03]. The bounds for the AKR algorithm are given in [Ger07, Lemma 3.4][6] and [Ger07, Theorem 4.1].

The problem that we study is a bit different. As our starting point, we take an algorithm that can approximate the q-power Frobenius matrix with a prescribed precision r. Then we *only* consider the precision that is lost due to calculating the characteristic polynomial.

Let us now discuss how to compute the characteristic polynomial of a matrix M that has entries in $\mathcal{V}$. It is obvious that this can be achieved using the naive formula for the determinant, without any loss of precision. But there is a better way to do this, using the following result of Vaccon.

Proposition 4.2.28. *Let $\mathcal{V}$ be a complete discrete valuation ring with local parameter π. Consider a square matrix M, with entries in $\mathcal{V}$, that is known*

[6]The formulation of this lemma is a bit confusing. It mentions the characteristic polynomial of the q-power Frobenius, yet it does not take into account that precision can be lost when computing this polynomial. In other words, the lemma only makes a statement about the *exact* characteristic polynomial. It is easy to reformulate this lemma using only the p-power and q-power Frobenius matrices.

with precision a. Assume that the determinant $\det(M) \in \mathcal{V}$ *has valuation* $b < a$. *Then one can compute the row-echelon form* $\widetilde{M}$ *of* M *with precision* $a - b$.

Proof. This is [Vac14, Theorem 3.2]. □

To calculate $\widetilde{M}$, one should use the Gaußian elimination algorithm with respect to pivots of minimal valuation. See [Vac14, Algorithm 1] for details. Since the row-echelon form $\widetilde{M}$ is known with precision $a - b$, the same is true for its determinant. We now show how this technique can be used to compute characteristic polynomials *without* loss of precision.

To this end we recall the *Gauß valuation* v_G on (the completion of) the field $K(T)$ of rational functions. On polynomials this valuation is defined by the equation

$$v_G(a_0 + a_1 T + \ldots + a_m T^m) = \min_j v_p(a_j)$$

with v_p the p-adic valuation on K.

Now consider a square matrix M with entries in $\mathcal{V} = W(\mathbb{F}_q)$. The polynomial

$$\det (T \cdot \mathrm{Id} - M) \tag{4.2.23}$$

is monic, so its Gauß valuation is zero. This means that the characteristic polynomial can be determined with precision r if the matrix M is known with precision r. This special case of proposition 4.2.28 is in fact very simple: we can always choose a quotient of monic polynomials as pivot, so we never divide by p.

The computation of (4.2.23) modulo p^r is also easy from a practical point of view. Indeed, the computations can be carried out in the ring of rational functions in T with coefficients in $\mathcal{V}/(p^r)$, whose denominators are not divisible by p. If we take $\mathcal{V} = \mathbb{Z}_p$ then this ring is very easy to represent in a computer.

Let us now look how this method can be applied to the characteristic polynomial from definition 4.2.21. By proposition 4.2.22 we only need to consider the matrix of the Frobenius action on $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}$. It is known that this matrix has entries in $p^{-\alpha}\mathcal{V}$, where α is defined by the formula:

$$\alpha = v_p((n-2)!) + \lfloor \log_p((n-2)!) \rfloor - v_p(q). \tag{4.2.24}$$

This bound is proved in [AKR11, Lemma 3.4.3], [AKR11, Proposition 3.4.6] and [Ger07, Lemma 3.3]. Also see [Ger07, Theorem 4.1].

The denominators in the Frobenius matrix lead to some additional loss of precision when one wishes to approximate the characteristic polynomial of Frobenius. However, since the denominators are bounded we easily obtain a bound on this loss of precision. We use this observation as the basis for an algorithm to compute the polynomials $P_m(T)$. See algorithm 4.3 below. For convenience we limit ourselves to the case $n \geq 3$.

Algorithm 4.3: Computing the approximate characteristic polynomial of Frobenius.

Data: A weighted homogeneous hypersurface singularity $Y = Z_{\mathbb{A}_k^n}(g)$ satisfying all the assumptions from definition 3.1.7 (together with a suitable lift $\mathcal{G}$ of g) and an integer $m \geq 1$.

Result: The polynomial $P_m(T) = P(T) \bmod p^m$, with $P(T)$ as in definition 4.2.21.

begin

Compute an integer α such that p^α times the Frobenius matrix of $H_{rig}^{n-1}(\mathbb{P}_k^{n-1} \setminus \widetilde{S}_\infty)^{G(\underline{w})}$ has entries in $\mathcal{V}$. One can take α as in equation (4.2.24).

Apply algorithm 4.1 to find a basis $v_1, v_2, \ldots, v_\delta$. Initialize an empty $\delta \times \delta$ matrix $\overline{M}$. Also define

$$r \coloneqq m + (\delta - 1) \cdot \alpha$$

Calculate an index N at which the sum (4.2.6) needs to be truncated in order to approximate the Frobenius matrix with precision r (or use the alternative technique from remark 4.2.27).

for $j \in \{1, \ldots, \delta\}$ **do**

Compute the sum (4.2.6) corresponding to the basis element v_j, truncated at the N^{th} term.

Apply algorithm 4.2 to compute the j-th column of $\overline{M}$.

end

Compute the polynomial

$$Q(T) \coloneqq \det(T \cdot \mathrm{Id} - p^\alpha \overline{M}).$$

using the approximate Gaußian elimination algorithm [Vac14, Algorithm 1].

Output:

$$p^{-\delta \cdot \alpha} \cdot Q(p^\alpha T) \bmod p^m.$$

end

Unfortunately the bound N at which to truncate the sum (4.2.6) quickly becomes unmanageable as the required precision r grows. The dependency of r on the dimension δ means that the algorithm is only practical when m and δ are both small.

However, in practice it often happens that the approximate matrix of Frobenius has no denominators. In this case proposition 4.2.28 can be used to compute the characteristic polynomial of Frobenius *without* loss of precision (i.e. we may take $r = m$).

Also note that the edge case $\delta = 1$ of algorithm 4.3 is consistent with proposition 4.2.23. If the dimension is equal to one then there cannot be any denominators in the Frobenius matrix, since $P(T) \in \mathcal{V}[T]$. In this case we can read off the characteristic polynomial $P_m(T)$ with m equal to the chosen precision in the modified AKR algorithm.

We now prove the correctness of algorithm 4.3.

Proposition 4.2.29. *The output of algorithm 4.3 is indeed equal to $P_m(T) = P(T)$ mod p^m, with $P(T)$ as in definition 4.2.21.*

Proof. We have proved earlier that the matrix $\overline{M}$ approximates the Frobenius matrix with precision r. It follows that $p^\alpha \overline{M}$ is an approximation of p^α times the Frobenius matrix with absolute precision $r + \alpha$. Moreover, we know that p^α times the Frobenius matrix has entries in $\mathcal{V}$. The method from [Vac14, Algorithm 1] can then be used to calculate the polynomial

$$Q(T) = \det(T \cdot \mathrm{Id} - p^\alpha \overline{M}).$$

According to proposition 4.2.28 this step does not lead to any loss of precision, therefore we know that $Q(T)$ is an approximation of precision $r + \alpha$ of the polynomial

$$\det \left(T \cdot \mathrm{Id} - p^\alpha \cdot \mathrm{Fr} \mid H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})} \right).$$

We now rewrite $P(T)$ as

$$P(T) = p^{-\delta \cdot \alpha} \cdot \det \left(p^\alpha \cdot T \cdot \mathrm{Id} - p^\alpha \cdot \mathrm{Fr} \mid H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})} \right).$$

Since we chose $r = m + (\delta - 1) \cdot \alpha$ it easily follows that

$$P_m(T) = p^{-\delta \cdot \alpha} \cdot Q(p^\alpha T) \bmod p^m.$$

This shows the correctness of algorithm 4.3. $\square$

Remark 4.2.30. It seems likely that algorithm 4.3 can be improved, see for example the suggestion in [AKR11, Remark 1.6.4]. This approach has been worked out in [Wal09] for the rigid cohomology of a smooth projective hypersurface minus a smooth hyperplane section. This solution also assumes that some of the constructions related to [Min13, Theorem B] can be made explicit. The improved bound for the precision loss that occurs when one calculates the characteristic polynomial is given in [Wal09, Theorem 6.1.5]. This bound no longer involves the dimension of the cohomology.

Exact calculation

To end this paragraph we discuss the finest possible invariant that can be computed with algorithm 4.3.

Let us first consider a homogeneous singularity (i.e. we assume $g = \widetilde{g}$). In this situation the local rigid cohomology $H^n_{rig,\{0\}}(Y)$ can be identified with the primitive part of $H^{n-2}_{rig}(\widetilde{S}_\infty)(-1)$. By following the proof of proposition 4.2.23 we then find that the characteristic polynomial $P(T)$ belongs to $\mathbb{Z}[T]$. This means that $P(T)$ can be recovered exactly by computing $P_{m_0}(T)$ for a certain value m_0 that is large enough.

By the Weil theorem, the eigenvalues λ of the Frobenius on $H^{n-2}_{rig}(\widetilde{S}_\infty)$ have complex norm $|\lambda| = q^{\frac{n-2}{2}}$. This provides upper bounds for the complex norms of the coefficients of $P(T)$, and in this way a precise value for m_0 can be determined. Compare this with paragraph 3 of [Ked01], where the same technique is applied to hyperelliptic curves.

In the general case it is a bit more difficult to see that $P(T)$ has integer coefficients. Using an unpublished result of Kloosterman[7] one can prove that the polynomial

$$Q(T) \coloneqq \varepsilon q^{-\beta} \cdot P(q^{n-1}T)$$

can be identified with a certain part of the zeta function of the weighted homogeneous hypersurface complement $\mathbb{P}_k(\underline{w}) \setminus S_\infty$. In this expression $\varepsilon \in \{\pm 1\}$ and the value $\beta \geq 0$ can be determined after a bit of computation. The polynomial $Q(T)$ belongs to $\mathbb{Z}[T]$ and as a result we have $P(T) \in \mathbb{Z}[1/p][T]$. Combining this with proposition 4.2.23 we find that $P(T) \in \mathbb{Z}[T]$ as claimed.

The local rigid cohomology of a weighted homogeneous singularity is related to the primitive part of $H^{n-2}_{rig}(\widetilde{S}_\infty)(-1)^{G(\underline{w})}$. One can again use the Weil theorem to find bounds on the complex norms of the coefficients of $P(T)$. Knowing that $P(T) \in \mathbb{Z}[T]$, these bounds can be used to find a value m_0 such that $P(T)$ can be deduced from $P_m(T)$ for $m \geq m_0$. In other words: the approximate polynomial $P_{m_0}(T)$ can be considered as the finest possible invariant that can be computed with algorithm 4.3.

4.3 Examples

In this section we discuss some specific *classes* of weighted homogeneous hypersurface singularities.

For the most part we will not rely on the algorithms from section 4.2. The reason is that these algorithms can only be applied to one example at a time, not to an entire class. Also, for some classes of singularities we are able to determine the characteristic polynomial of Frobenius *exactly*, something which is difficult to achieve using the modified AKR algorithm.

We start this section by proving a simple technique that will allow us to study several types of singularities. We apply this technique to ordinary double points, singularities of type A_j and unimodal singularities. We end this

[7] See proposition 5.1.6 in the next chapter.

section with some general remarks about weighted homogeneous singularities on curves.

Perhaps the most important reason for discussing these examples is that they provide test cases for our implementation of the modified AKR algorithm.

In this section we use the notation $H^\bullet$ as an abbreviation for $H^\bullet_{rig}$.

4.3.1 Counting points on a normal form

We start by describing a simple but effective technique that is specific to weighted homogeneous singularities.

Proposition 4.3.1. *Consider a field $k = \mathbb{F}_q$ and let $g, g' \in k[x_1, \ldots, x_n]$ be two weighted homogeneous polynomials such that the singularities $Y = \mathbb{Z}_{\mathbb{A}^n_k}(g)$ and $Y' = \mathbb{Z}_{\mathbb{A}^n_k}(g')$ satisfy all the conditions of definition 3.1.7. Also assume that $n \geq 3$.*

If there exists a Frobenius-equivariant isomorphism $H^n_{\{0\}}(Y) \cong H^n_{\{0\}}(Y')$ then $|Y(k)| = |Y'(k)|$.

If moreover $\dim H^n_{\{0\}}(Y) = \dim H^n_{\{0\}}(Y') = 1$ then the converse property holds.

Proof. Recall from paragraph 3.2 that we have Frobenius-equivariant isomorphisms $H^i_{\{0\}}(Y) \cong H^i(\mathbb{A}^n_k \setminus Y)(+1)$ for $2 \leq i \leq n$. By combining this with theorem 3.1.11 we find that $H^{n-1}(\mathbb{A}^n_k \setminus Y) \cong H^n(\mathbb{A}^n_k \setminus Y)(+1)$ and $H^i(\mathbb{A}^n_k \setminus Y) = 0$ for $2 \leq i \leq n-2$.

In the proof of proposition 3.2.1 we have also shown that there is an isomorphism $H^1(\mathbb{A}^n_k \setminus Y) \cong H^0(Y \setminus \{0\})(-1)$. It follows that $H^1(\mathbb{A}^n_k \setminus Y)$ is 1-dimensional with the Frobenius acting as multiplication by q. As usual, $H^0(\mathbb{A}^n_k \setminus Y)$ is 1-dimensional with Frobenius acting as the identity.

We can combine all this with the trace formula (1.2.1) for the scheme $\mathbb{A}^n_k \setminus Y$, which is smooth affine. By doing so we obtain

$$\begin{aligned}
|Y(k)| &= q^n - |(\mathbb{A}^n_k \setminus Y)(k)| \\
&= q^n - \sum_{i=0}^{n} (-1)^i \mathrm{Tr}(q^n \mathrm{Fr}^{-1} \mid H^i(\mathbb{A}^n_k \setminus Y)) \\
&= q^n - [q^n - q^{n-1} + \sum_{i=2}^{n} (-1)^i \mathrm{Tr}(q^{n-1} \mathrm{Fr}^{-1} \mid H^i_{\{0\}}(Y)) \\
&= q^n - [q^n - q^{n-1} + (-1)^{n-1} q^n T + (-1)^n q^{n-1} T] \\
&= q^{n-1} + (-1)^n q^n T + (-1)^{n+1} q^{n-1} T
\end{aligned}$$

where $T = \mathrm{Tr}(\mathrm{Fr}^{-1} \mid H^n_{\{0\}}(Y))$. By a similar argument we have

$$|Y'(k)| = q^{n-1} + (-1)^n q^n T' + (-1)^{n+1} q^{n-1} T'$$

with $T' = \mathrm{Tr}(\mathrm{Fr}^{-1} \mid H^n_{\{0\}}(Y'))$. If $H^n_{\{0\}}(Y)$ and $H^n_{\{0\}}(Y')$ are Frobenius-isomorphic then it follows that $|Y(k)| = |Y'(k)|$, as required.

Now assume that $\dim H^n_{\{0\}}(Y) = \dim H^n_{\{0\}}(Y') = 1$. In this case we have $T = F^{-1}$ and $T' = (F')^{-1}$ with $F, F' \in K$ the Frobenius matrix of $H^n_{\{0\}}(Y)$ resp. of $H^n_{\{0\}}(Y')$. If $|Y(k)| = |Y'(k)|$ then it follows that $T = T'$ or $F = F'$. This finishes the proof. $\square$

This result gives a very simple way to distinguish non-equivalent weighted homogeneous singularities, at least when q and n are small.

Note however that the number of rational points on the normal form should not be considered as a true invariant, since this number only makes sense for a weighted homogeneous singularity. The invariants from sections 4.1 and 4.2 on the other hand are defined for any isolated singularity. Our goal in the previous two sections was to show that these invariants happen to be *computable* for weighted homogeneous singularities.

Distinguishing forms over $\overline{k}$

So far we have described several methods to distinguish non-equivalent weighted homogeneous forms over a finite base field $\mathbb{F}_{p^r}$. Indeed, suppose that $g, g' \in \mathbb{F}_{p^r}[x_1, \ldots, x_n]$ are weighted homogeneous polynomials whose dimension, point-count or approximate characteristic polynomial of Frobenius are different. Then we can conclude that g and g' are not contact equivalent over $\mathbb{F}_{p^r}$.

This algorithmic approach is less suited for distinguishing forms over $\overline{\mathbb{F}_p}$. If we are given weighted homogeneous polynomials $g, g' \in \overline{\mathbb{F}_p}[x_1, \ldots, x_n]$ then there exists of course an r such that g and g' are defined over $\mathbb{F}_{p^r}$. However, it is possible that g and g' are contact equivalent over $\overline{\mathbb{F}_p}$ but not over $\mathbb{F}_{p^r}$. An easy example of this phenomenon can be constructed using proposition 4.3.4 in the next paragraph. So if g and g' have different point-counts or different approximate characteristic polynomials over $\mathbb{F}_{p^r}$, then g and g' may still be contact equivalent over $\overline{\mathbb{F}_p}$.

What we do know is that if g and g' are contact equivalent over $\overline{\mathbb{F}_p}$ then there exists a finite extension $\mathbb{F}_q \supset \mathbb{F}_{p^r}$ such that g and g' are contact equivalent over $\mathbb{F}_q$. So if one can distinguish $g, g' \in \mathbb{F}_{p^r}[x_1, \ldots, x_n]$ over any finite extension $\mathbb{F}_q \supset \mathbb{F}_{p^r}$, then g and g' cannot be contact equivalent over $\overline{\mathbb{F}_p}$. This property can (in principle) be tested by replacing the point-counting technique of proposition 4.3.1 with a comparison of the zeta functions of $Y = Z_{\mathbb{A}^n_k}(g)$ and $Y' = Z_{\mathbb{A}^n_k}(g')$. These zeta functions can theoretically be computed using the algorithms from section 4.2. With this technique it should be possible to distinguish forms over $\overline{\mathbb{F}_p}$. However, it requires a bit more effort to develop this idea into a general algorithm.

It seems that the most practical way to distinguish forms over $\overline{\mathbb{F}_p}$ is still the formal-analytical approach that is used in [GK90] and in many other papers by Greuel et al. This technique works over *any* algebraically closed base field, but not over a finite base field.

4.3.2 Ordinary double points

The next step is to apply a point-counting technique to the study of ordinary double points. We start by defining this class of singularities.

Definition 4.3.2. Consider a field $k = \mathbb{F}_q$ with q odd and denote by $\mathfrak{M}$ the maximal ideal $(x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$. A singularity $Y = Z_{\mathbb{A}^n_k}(g)$ with $g \in \mathfrak{M}^2$ is said to be an *ordinary double point* if the Hessian matrix

$$\mathrm{Hess}(g) = \left(\frac{\partial^2 g}{\partial x_i \partial x_j}(0) \right)_{1 \le i,j \le n}$$

has full rank.

We verify that this definition is stable under contact equivalence. This means that the definition above really describes a class of singularities, not just a class of equations.

Proposition 4.3.3. *Use the same notations as in definition 4.3.2. If two polynomials $g, g' \in \mathfrak{M}^2$ are contact equivalent then Hess(g) and Hess(g') have equal rank.*

Proof. Note that for $g \in \mathfrak{M}^2$, the matrix Hess(g) only depends on g modulo $\mathfrak{M}^3$. We can then write

$$g \equiv (x_1, \ldots, x_n)\, M\, (x_1, \ldots, x_n)^t \bmod \mathfrak{M}^3$$

with M a matrix with entries in k. Since $\mathrm{char}(k) \neq 2$ we may assume that M is symmetric. A direct calculation shows that $\mathrm{Hess}(g) = 2M$. In other words,

$$\operatorname{rank} \mathrm{Hess}(g) = \operatorname{rank} M.$$

Since by assumption $g \sim_c g'$ there exists an automorphism φ of $k[[x_1, \ldots, x_n]]$ such that $\varphi(g) = ug'$ with u a unit. Now write $f_i = \varphi(x_i)$ for every $1 \le i \le n$. By using the inverse function theorem [GP08, Theorem 6.2.18] we see that the matrix that is formed by the linear parts of the f_i is invertible. This means that that there is an invertible matrix N such that

$$(f_1, \ldots, f_n)^t \equiv N(x_1, \ldots, x_n)^t \bmod \mathfrak{M}^2.$$

Now we have that

$$\begin{aligned} ug' = \varphi(g) &\equiv (f_1, \ldots, f_n)\, M\, (f_1, \ldots, f_n)^t \bmod \mathfrak{M}^3 \\ &\equiv (x_1, \ldots, x_n)\, N^t M N\, (x_1, \ldots, x_n)^t \bmod \mathfrak{M}^3 \end{aligned}$$

Since M and $N^t M N$ have equal rank we see that

$$\operatorname{rank} \operatorname{Hess}(g) = \operatorname{rank} \operatorname{Hess}(ug').$$

Also, since $g' \in \mathfrak{M}^2$ and since the constant term of u is nonzero we have that $ug' \equiv g' \mod \mathfrak{M}^3$. Therefore we also have $\operatorname{rank} \operatorname{Hess}(ug') = \operatorname{rank} \operatorname{Hess}(g')$. This finishes the proof. □

Next we have the well-known *Morse lemma*, which gives us the canonical equation for an ordinary double point.

Proposition 4.3.4. *Again assume that char$(k) \neq 2$ and that $g \in \mathfrak{M}^2$. Then $Y = Z_{\mathbb{A}^n_k}(g)$ is an ordinary double point if and only if g is contact equivalent to a form*

$$\alpha_1 x_1^2 + \ldots + \alpha_n x_n^2$$

for certain $\alpha_1, \ldots, \alpha_n \in k^\times$.

Proof. Assume that $\operatorname{Hess}(g)$ is invertible. We can write

$$g = \sum_{i,j} x_i x_j H_{ij}(x_1, \ldots, x_n) = \underline{x} H(\underline{x}) \underline{x}^t$$

where H is an $n \times n$ matrix with polynomial entries and $\underline{x} = (x_1, \ldots, x_n)$. Since $\operatorname{char}(k) \neq 2$ we may assume that H is symmetric. A simple computation then shows that

$$\operatorname{Hess}(g) = 2 \cdot H(0).$$

It follows that the matrix $H(0)$ is invertible. Now let $M \in \mathrm{GL}_n(k)$ be such that $MH(0)M^t$ is a diagonal matrix. Such an M always exists for a base field of characteristic $\neq 2$. By applying the automorphism $\underline{x} \mapsto \underline{x}M^{-1}$ of $\mathbb{A}^n_k$ we reduce to the case where $H(0)$ is a diagonal matrix.

Now we need to construct an automorphism of $k[[x_1, \ldots, x_n]]$ that transforms g into the required form. In informal terms this boils down to "completing the squares". In order to do this we need to be able to take square roots of formal power series, at least under certain conditions. The main technical ingredient here is a version of Hensel's lemma. More specifically, we refer to [GP08, Exercise 6.2.5].

The details of the calculation are almost the same as over the complex numbers, see for example [dJP00, Lemma 3.4.30]. The difference is that the implicit function theorem and the inverse function theorem should be replaced by Hensel's lemma resp. by [GP08, Theorem 6.2.18]. □

Remark 4.3.5. It can be verified that if the matrix $H(0)$ in the proof above is diagonal, then $\alpha_i = H_{ii}(0)$ for every $1 \leq i \leq n$. The proof in example 3.1.3 can then be completed by calculating the Hessian matrix of equation (3.1.2).

We find that each singular point on Schoen's quintic is contact equivalent to the equation $b_1\, y_1^2 + \ldots + b_4\, y_4^2$.

We now use the canonical equation of an ordinary double point to completely determine the Frobenius action on its local rigid cohomology.

Proposition 4.3.6. *We work over a finite field $k = \mathbb{F}_q$ with q odd. Consider for $n \geq 3$ the form*

$$g = \alpha_1\, x_1^2 + \ldots + \alpha_n\, x_n^2$$

with $\alpha_i \neq 0$ for all i. Write $Y = Z(g) \subset \mathbb{A}_k^n$. Then the dimension of the local rigid cohomology $H^n_{\{0\}}(Y)$ is equal to

$$\begin{cases} 1 & \textit{for } n \textit{ even} \\ 0 & \textit{for } n \textit{ odd} \end{cases}$$

If n is even then the Frobenius is equal to the multiplication by $\varepsilon q^{\frac{n}{2}}$, where

$$\varepsilon = \eta((-1)^{\frac{n}{2}} \alpha_1 \ldots \alpha_n)$$

with η the quadratic character of $\mathbb{F}_q^\times$.

Proof. By theorem 3.1.11 we have an isomorphism

$$H^n_{\{0\}}(Y) \xrightarrow{\sim} H^{n-1}(\mathbb{P}_k^{n-1} \setminus \widetilde{S}_\infty)$$

where $\widetilde{S}_\infty = Z(g) \subset \mathbb{P}_k^{n-1}$. The dimension of the right-hand side is equal to $\frac{d-1}{d}((d-1)^{n-1} + (-1)^n)$ with $d = 2$. This proves the first part of the proposition.

For n even the Frobenius on the local cohomology is given by the multiplication with a constant $F \in K$. By using the same calculations as in the proof of proposition 4.3.1 we find that

$$|Y(k)| = q^{n-1} + F^{-1}(q-1)q^{n-1}.$$

On the other hand, it is a classical fact that

$$|Y(k)| = q^{n-1} + \varepsilon(q-1)q^{\frac{n-2}{2}}$$

with ε as in the statement of the proposition. See [LN97, Theorem 6.26] for a proof. It easily follows that $F = \varepsilon q^{\frac{n}{2}}$. $\square$

Remark 4.3.7. Proposition 4.3.6 provides a way to test our implementation of the algorithms in section 4.2. We have tried out several examples of ordinary double points with n even. The answer was always compatible with proposition 4.3.6.

4.3.3 Singularities of type A_j

As our next example we consider the singularities of type A_j.

Definition 4.3.8. Fix an integer $j \geq 1$ and define

$$m = \begin{cases} \frac{j+1}{2} & \text{if } j \text{ is odd} \\ j+1 & \text{if } j \text{ is even} \end{cases}$$

Now consider a finite base field k. We say that a hypersurface singularity on a k-scheme is *of type A_j* if it is contact equivalent to the singularity with local equation

$$g = x_1^{j+1} + x_2^2 + \ldots + x_n^2. \tag{4.3.1}$$

Remark 4.3.9. The conditions of theorem 3.1.11 are satisfied if the characteristic of k does not divide $2m$ and if $n \geq 3$. In the rest of this paragraph we work under these two assumptions. We also write $Y = Z_{\mathbb{A}_k^n}(g)$ with g as in (4.3.1).

Note that according to our definition 4.3.2 and proposition 4.3.4, an ordinary double point need *not* be of type A_1.

Clearly g is weighted homogeneous with weights

$$\underline{w} = \begin{cases} (1, m, \ldots, m) & \text{if } j \text{ is odd} \\ (2, m, \ldots, m) & \text{if } j \text{ is even} \end{cases}$$

We start by examining the dimension of an A_j singularity. For this we may interpret (4.3.1) as an equation with coefficients in $\mathbb{C}$. After invoking theorem 3.1.11 and applying [Gro66, Theorem 1] we find that

$$\dim_K H^n_{rig,\{0\}}(Y) = \dim_{\mathbb{C}} H^{n-1}(\mathbb{P}_{\mathbb{C}}(\underline{w}) \setminus S_\infty, \mathbb{C})$$

where $S_\infty = Z(g) \subset \mathbb{P}_{\mathbb{C}}(\underline{w})$. We now apply a remark from [Dol82], which says that a weighted projective space $\mathbb{P}_{\mathbb{C}}(\underline{w})$ with weights satisfying

$$\gcd(w_1, \ldots, w_n) = 1$$

is isomorphic to a certain other space $\mathbb{P}_{\mathbb{C}}(\underline{w'})$ with weights satisfying

$$\gcd(w'_1, \ldots, \widehat{w'_i}, \ldots, w'_n) = 1$$

for every i. There are two cases to consider.

Case I: j is odd

In this case there is an isomorphism $\mathbb{P}_{\mathbb{C}}(\underline{w}) \cong \mathbb{P}_{\mathbb{C}}^{n-1}$. The precise calculation is as in paragraph 1.3 of [Dol82]. The equation (4.3.1) is transformed into

$g' = x_1^2 + \ldots + x_n^2$. Therefore we have

$$\dim_K H^n_{rig,\{0\}}(Y) = \dim_{\mathbb{C}} H^{n-1}(\mathbb{P}^{n-1}_{\mathbb{C}} \setminus Z(g'),\, \mathbb{C}).$$

We have seen in the proof of proposition 4.3.6 that this space is one-dimensional for n even and zero-dimensional for n odd.

Case II: j is even

We claim that in this case $\dim_K H^n_{rig,\{0\}}(Y) = 0$. To see this we first apply proposition 3.4.1. With $S_\infty = Z(g) \subset \mathbb{P}_{\mathbb{C}}(\underline{w})$ as above we find that

$$H^{n-1}(\mathbb{P}_{\mathbb{C}}(\underline{w}) \setminus S_\infty,\, \mathbb{C}) \xrightarrow{\sim} H^{n-1}(S \setminus S_\infty,\, \mathbb{C})^{\langle \zeta_d \rangle}. \tag{4.3.2}$$

Since we are working over $\mathbb{C}$ the monodromy action on $S \setminus S_\infty$ that we defined in paragraph 3.3.1 corresponds to the classical analytic monodromy. A proof of this fact can be found in chapter 3 of [Dim92]. We can now apply the Sebastiani-Thom theorem (c.f. the main result of [ST71]) to the form (4.3.1). It is mentioned on the last page of [ST71] that the monodromy associated to the form $u_1^2 + u_2^2$ is the identity map on a one-dimensional space. By combining the theorem of [ST71] with equation (4.3.2) we reduce to the case where

$$g = x_1^{j+1} + x_2^2 + x_3^2 \quad \text{or} \quad g = x_1^{j+1} + x_2^2 + x_3^2 + x_4^2.$$

By applying the same weight-change technique as in the previous case we find an isomorphism $\mathbb{P}_{\mathbb{C}}(\underline{w}) \cong \mathbb{P}_{\mathbb{C}}(\underline{w'})$ where $\underline{w'} = (2, 1, \ldots, 1)$. Under this isomorphism our equation g is transformed to

$$g' = x_1 + x_2^2 + x_3^2 \quad \text{resp.} \quad g' = x_1 + x_2^2 + x_3^2 + x_4^2.$$

The dimension of $H^{n-1}(\mathbb{P}_{\mathbb{C}}(\underline{w'}) \setminus Z(g'),\, \mathbb{C})$ can now be computed using proposition 4.1.1. Both possibilities for g' give dimension zero.

These results about the dimension of an A_j singularity are compatible with [Dim90a, Example 1.9]. This paper also considers the mixed Hodge structure on $H^n_{\{0\}}(Z(g),\, \mathbb{C})$ for n even and g the complex equation (4.3.1) with j odd. It is shown that these mixed hodge structures are all the same, namely of type $(\frac{n}{2}, \frac{n}{2})$. This suggests that the Frobenius on the local rigid cohomology of an A_j singularity with j odd should only depend on n and on the cardinality of the base field. We show that this is indeed the case.

Proposition 4.3.10. *Take j odd and $n \geq 4$ even. Also fix a base field $k = \mathbb{F}_q$ with q odd. Then the Frobenius on the n-th local cohomology of a form of type A_j is given by the multiplication with*

$$(\eta(-1) \cdot q)^{\frac{n}{2}},$$

where η denotes the quadratic character on $\mathbb{F}_q$.

Proof. Again we use the isomorphism $\mathbb{P}_k(\underline{w}) \cong \mathbb{P}_k^{n-1}$, but this time over the finite base field $k = \mathbb{F}_q$. Under this isomorphism, the hypersurface $S_\infty = Z(g) \subset \mathbb{P}_k(\underline{w})$ is transformed into $Z(g') \subset \mathbb{P}_k^{n-1}$, where

$$g' = x_1^2 + \ldots + x_n^2.$$

In particular we see that the complement $\mathbb{P}_k(\underline{w}) \setminus S_\infty \cong \mathbb{P}_k^{n-1} \setminus Z(g')$ is smooth. It follows that the canonical map

$$H_{rig}^{n-1}(\mathbb{P}_k(\underline{w}) \setminus S_\infty) \longrightarrow H_{rig}^{n-1}(\mathbb{P}_k^{n-1} \setminus \widetilde{S}_\infty)^{G(\underline{w})} \tag{4.3.3}$$

is injective. Indeed, this is a special case of point i) of proposition 3.3.6.

By theorem 3.1.11, the right hand-side of (4.3.3) can be identified with the local rigid cohomology space $H^n_{rig,\{0\}}(Y)$, with $Y = Z(g) \subset \mathbb{A}_k^n$. We have computed before that its dimension is equal to one. Similarly, and by using the isomorphism $\mathbb{P}_k(\underline{w}) \setminus S_\infty \cong \mathbb{P}_k^{n-1} \setminus Z(g')$, the left-hand side of (4.3.3) can be identified with the local rigid cohomology of an ordinary double point. By proposition 4.3.6 we see that the left-hand side of (4.3.3) also has dimension one.

It follows that the map (4.3.3) is a Frobenius-equivariant isomorphism. The formula for the Frobenius action on $H^n_{rig,\{0\}}(Y)$ now follows immediately from proposition 4.3.6. $\square$

Remark 4.3.11. In the proof of proposition 4.3.10 we have shown that the canonical map (4.2.20) is an isomorphism when S_∞ is defined by the equation of an A_j-singularity. However, this situation is special because $\mathbb{P}_k(\underline{w}) \setminus S_\infty$ happens to be smooth. In general, the weighted projective complement $\mathbb{P}_k(\underline{w}) \setminus S_\infty$ has singularities, and proposition 3.3.6 is not applicable. In this situation we are not aware of any proof that the map (4.2.20) is injective. We also refer to section 5.1, where the weight-change technique will be described in general.

4.3.4 Unimodal singularities

In this paragraph we consider the class of *unimodal singularities* on a surface (i.e., we take $n = 3$). Over $\mathbb{C}$ there are two equivalent sets of equations that one can use to define this class. In [Bri79] a unimodal singularity is defined as a hypersurface singularity that is contact equivalent to one of the weighted homogeneous forms

$$\begin{array}{lcl} T_{3,3,3} & : & x_1^3 + x_2^3 + x_3^3 + c \cdot x_1 x_2 x_3 \\ T_{2,4,4} & : & x_1^2 + x_2^4 + x_3^4 + c \cdot x_1 x_2 x_3 \\ T_{2,3,6} & : & x_1^2 + x_2^3 + x_3^6 + c \cdot x_1 x_2 x_3 \end{array} \tag{4.3.4}$$

In [Sai74] a unimodal singularity is defined by the set of weighted homogeneous equations[8]

$$\begin{array}{rcl} \widetilde{E}_6 & : & x_2(x_2-x_1)(x_2-\lambda\cdot x_1)-x_1x_3^2 \\ \widetilde{E}_7 & : & x_2x_1(x_2-x_1)(x_2-\lambda\cdot x_1)-x_3^2 \\ \widetilde{E}_8 & : & x_2(x_2-x_1^2)(x_2-\lambda\cdot x_1^2)-x_3^2 \end{array} \tag{4.3.5}$$

Suppose now that we work over a finite base field $k=\mathbb{F}_q$ with $\mathrm{char}(k)\notin\{2,3\}$. It is easy to verify that the projective hypersurfaces $\widetilde{S}_\infty$ associated to the forms (4.3.4) and (4.3.5) are "almost always" smooth. Only a finite number of characteristics for the ground field k need to be excluded (for a fixed choice of the parameters c and λ).

To see this, we consider (4.3.4) and (4.3.5) as elements of $\mathbb{Q}[x_1,x_2,x_3]$, and we compute the reduced Gröbner bases of the Jacobian ideals. We will also fix the parameter values $c=1$ and $\lambda=2$. Using the lexicographic monomial order and the convention $x_3\preccurlyeq_{lex}x_2\preccurlyeq_{lex}x_1$, we find that each Gröbner basis contains a power of one or more variables, which means that these variables can be eliminated. With this information it is easy to check that the hypersurfaces $\widetilde{S}_\infty$ associated to equations (4.3.4) and (4.3.5), viewed over $\mathbb{Q}$, are smooth. The same property must then hold over a base field k of positive characteristic, if we exclude a finite number of characteristics. A direct calculation shows that the conditions of definition 3.1.7 are verified when $\mathrm{char}(k)\in\{5,11,13\}$.

We can then use proposition 4.1.2 to verify that the local rigid cohomology of a unimodal singularity over $\mathbb{F}_q$ has dimension 2, regardless of whether we use the equations (4.3.4) or (4.3.5). This is compatible with [Dim90a, Example 1.10]. However, in [Dim90a] it is also shown that the complex equations (4.3.4) and (4.3.5) all give rise to the same type of mixed Hodge structure on the local cohomology. By analogy we expect that the equations that define a unimodal singularity over $\mathbb{F}_q$ have isomorphic Frobenius structures, at least for a fixed choice of the parameter. We can test this property using proposition 4.3.1. The results are presented in table 4.1 (again fixing $c=1$ and $\lambda=2$). We see

p	$T_{3,3,3}$	$T_{2,4,4}$	$T_{2,3,6}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$
5	25	33	21	33	33	33
11	121	81	171	121	121	121
13	217	193	193	97	97	97

Table 4.1: Point-counts of equations (4.3.4) with $c=1$ and (4.3.5) with $\lambda=2$ over $\mathbb{F}_p$.

that the equations (4.3.4) cannot have isomorphic Frobenius structures. On the other hand we can use the algorithms from section 4.2 to verify that the equations (4.3.5) all give rise to the same characteristic polynomial of Frobe-

[8]Certain values for the parameters c and λ must be excluded. See the cited papers for details.

nius, at least modulo some power of p. We have calculated the characteristic polynomials with high precision for $p \in \{5, 11, 13\}$ and for each equation $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ (with $\lambda = 2$). It seems that for a fixed p, the modified AKR algorithm will converge to the same characteristic polynomial, for each equation in (4.3.5). These (conjectural) characteristic polynomials are displayed in table 4.2. We have carried out the same experiment for different values of the parameters c and λ, and each time we obtained similar results. This suggests that the equations (4.3.5) give the "right" definition for unimodal singularities over a finite base field. More precisely: the equations (4.3.5) seem to have the expected properties over a given base field $\mathbb{F}_q$, whereas the equations (4.3.4) will only enjoy these properties over an extension $\mathbb{F}_{q^a}$.

p	Characteristic polynomial of (4.3.5)
5	$T^2 + 2 \cdot 5\,T + 5^3$
11	$T^2 + 0 \cdot T + 11^3$
13	$T^2 - 6 \cdot 13\,T + 13^3$

Table 4.2: Characteristic polynomials of Frobenius for the equations (4.3.5) with $\lambda = 2$, over the field $\mathbb{F}_p$.

4.3.5 The case $n = 2$

We end this section with a remark about weighted homogeneous singularities on curves. This corresponds to taking $n = 2$. We show that for such singularities the characteristic polynomial of Frobenius always has a specific form.

Proposition 4.3.12. *Fix a finite base field $k = \mathbb{F}_q$. Let $g \in k[x_1, x_2]$ be weighted homogeneous of degree d w.r.t. weights $\underline{w} = (w_1, w_2)$ and consider the singularity $Y = Z_{\mathbb{A}^2_k}(g)$. Assume that this singularity satisfies all the conditions of definition 3.1.7. Then the characteristic polynomial of the Frobenius on $H^2_{rig,\{0\}}(Y)$ is of the form*

$$P(T) = \prod_i (T - q \cdot \alpha_i) \tag{4.3.6}$$

where the α_i are unit roots of order $\leq d$. If moreover $\underline{w} = (1, 1)$ then

$$P(T) = \prod_i (T^{d_i} - q^{d_i}) \tag{4.3.7}$$

where the d_i are the degrees of the irreducible factors of g.

Proof. According to proposition 4.2.22 we can write the characteristic polynomial of Frobenius as $(T - q)$ times the characteristic polynomial of the

Frobenius on $H^1(\mathbb{P}^1_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}$. It is then also clear that we only need to consider the case $\underline{w} = (1,1)$. Indeed, any monic polynomial that divides (4.3.7) is of the form (4.3.6).

So we assume that g is homogeneous. Let $g = \prod_i g_i$ be the factorization into irreducible factors, with each g_i being of degree d_i. By the Chinese remainder theorem we have

$$\widetilde{S}_\infty = \operatorname{Proj}\left(\frac{k[x_1,x_2]}{g(x_1,x_2)}\right) \cong \coprod_i \operatorname{Proj}\left(\frac{k[x_1,x_2]}{g_i(x_1,x_2)}\right).$$

The scheme $\operatorname{Proj}\left(\frac{k[x_1,x_2]}{g_i(x_1,x_2)}\right)$ has exactly one closed point, which is of degree d_i. Therefore its zeta function is equal to

$$\exp\left(\sum_{j=1}^{\infty}\frac{T^{j\cdot d_i}}{j}\right) = \exp(-\log(1-T^{d_i})) = (1-T^{d_i})^{-1}.$$

From this we obtain the identity

$$Z(\mathbb{P}^1_k \setminus \widetilde{S}_\infty, T) = \frac{\prod_i (1-T^{d_i})}{(1-T)(1-qT)}.$$

Now we can use the formula (1.2.2) that relates the zeta function to Monsky-Washnitzer cohomology. Since $H^0(\mathbb{P}^1_k \setminus \widetilde{S}_\infty)$ is one-dimensional with Frobenius acting as the identity, we must have

$$\det\left(1 - q\operatorname{Fr}^{-1}T \mid H^1(\mathbb{P}^1_k \setminus \widetilde{S}_\infty)\right) = (1-T)^{-1}\cdot\prod_i(1-T^{d_i}).$$

The formula for $P(T)$ easily follows. □

Remark 4.3.13. A polynomial g satisfying the conditions of definition 3.1.7 can only be reducible when $n = 2$. Indeed, for $n \geq 3$ the smoothness condition on $\widetilde{S}_\infty$ implies that $\widetilde{g}$, hence also g, is irreducible.

Remark 4.3.14. Proposition 4.3.12 gives another test case for our implementation of the the algorithms from section 4.2. Consider for example $g = x_1^3 + x_2^3$ over a prime field $\mathbb{F}_p$ with $p \neq 2, 3$. If $p \equiv 1 \bmod 3$ then $\mathbb{F}_p$ has a square root of -3. In this case g has three irreducible factors. In the other case g has exactly two irreducible factors. In both cases we know the characteristic polynomial of Frobenius.

Chapter 5

Some global questions

In this last chapter we discuss some open problems that are of a global nature.

We start with a survey about what is (not) known about the commuting of cohomology with finite groups. In chapter 3 we have used the fact that rigid cohomology commutes with finite groups under certain specific conditions. In section 5.1 we consider the big picture, and we describe what type of commuting properties are to be expected in general.

In the rest of this chapter we discuss some questions about the global rigid cohomology of a singular hypersurface. More specifically, we consider a hypersurface $X \subset \mathbb{P}^n_k$ having only weighted homogeneous singularities. Throughout this chapter we will also fix a global homogeneous equation $F \in k[x_0, x_1, \ldots, x_n]$ for this hypersurface. Then we investigate what can be said about the rigid cohomology $H^\bullet_{rig}(\mathbb{P}^n_k \setminus X)$.

Theorem 3.1.11 gives us a fairly explicit description of the local rigid cohomology spaces $H^\bullet_{rig,\{x\}}(X)$, where $x \in X$ is a singular point. One can ask how these local objects are related to the *global* object $H^\bullet_{rig}(\mathbb{P}^n_k \setminus X)$. For Betti cohomology over $\mathbb{C}$, one such relation has been proved by Dimca in [Dim90a], using certain topological techniques.

In section 5.2 we investigate whether the techniques of Dimca may be adapted to study the rigid cohomology $H^\bullet_{rig}(\mathbb{P}^n_k \setminus X)$. We will show that some intermediate results do carry over to rigid cohomology. This is due to certain vanishing properties that we will discuss in section 5.3. However, it seems that the main result from [Dim90a] cannot be easily translated to rigid cohomology.

Section 5.4 contains a possible corollary of the overconvergent site, namely a long exact sequence associated to a resolution of singularities $\widetilde{X} \to X$. Such a sequence would make it possible to study the rigid cohomology of X in terms of the smooth scheme $\widetilde{X}$, at least if one finds a suitable resolution. We argue that the only point that is missing for the proof is a *proper base change theorem* on the overconvergent site.

5.1 Commuting of rigid cohomology with finite groups

Consider a k-scheme X and a finite group $G \subset \mathrm{Aut}(X)$ that acts on X from the right. Under some mild assumptions on X we can form the quotient X/G, see paragraph 3.3.2 for details. From the quotient map $X \to X/G$ we then obtain a map

$$H^\bullet_{rig}(X/G) \longrightarrow H^\bullet_{rig}(X)^G \tag{5.1.1}$$

on rigid cohomology. A crucial fact that we used in chapter 3 is that the map (5.1.1) is an isomorphism when the quotient map $X \to X/G$ is a finite étale Galois cover and X/G is smooth. It is natural to ask if this property holds more generally.

Question 5.1.1. *When is the map (5.1.1) an isomorphism?*

The sufficient condition that $X \to X/G$ is an étale cover was proved by Etesse in [Ete08, Théorème IV.4.2]. This property is to be expected, since it can be seen as an analogue of the *Hochschild-Serre spectral sequence* for étale cohomology. See [Mil80, Theorem III.2.20] for the precise formulation. This theorem is quite general, in that it provides a relation between the cohomology of any étale sheaf $\mathcal{F}$ on X/G and the cohomology of its restriction to X. However, applying this result to the ℓ-adic cohomology $H^\bullet_{\acute{e}t}(X/G, \mathbb{Q}_\ell)$ yields an isomorphism similar to (5.1.1).

One limitation of [Ete08, Théorème IV.4.2] is that the quotient X/G is assumed to be smooth. It seems plausible that this assumption is not essential, since the isomorphism derived from [Mil80, Theorem III.2.20] holds for any finite étale Galois cover. This leads to the following conjecture.

Conjecture 5.1.2. *The map (5.1.1) is an isomorphism whenever the quotient $X \to X/G$ is a finite étale Galois cover.*

A proof of this conjecture could possibly be derived from the special case [Ete08, Théorème IV.4.2] by using cohomological descent. Indeed, consider a proper hypercover $Y_\bullet \to X/G$ such that each scheme Y_i is smooth. Such a hypercover is guaranteed to exist by the main theorem of [dJ96][1]. Each pullback $X_i = Y_i \times_{X/G} X$ is of course an étale Galois cover of Y_i. Also, the X_i form a proper hypercover $X_\bullet \to X$. If one knows that each cover $X_i \to Y_i$ has the same Galois group G then conjecture 5.1.2 follows by applying cohomological descent. However, this last claim would require some more effort to prove.

[1]The generalization [dJ96, Theorem 7.3] can be used to construct a proper hypercover $X_\bullet \to X$ where each X_i is equipped with a G-action, and every morphism $X_i \to X_{i-1}$ is G-equivariant. However, it is not clear that the quotients X_i/G have any relation with the Y_i

5.1.1 The situation in characteristic zero

It seems difficult to give a general answer to question 5.1.1. Except for étale Galois covers we are not aware of any classes of G-actions for which such a property holds. However, for the Betti cohomology of $\mathbb{C}$-schemes the commuting with G-actions is a well-known fact.

Proposition 5.1.3. *Consider a separated $\mathbb{C}$-scheme X with an action of a finite group G. Assume that X can be covered by G-stable affine opens, so that the quotient X/G exists. Then the quotient $X \to X/G$ induces an isomorphism*

$$H^\bullet(X/G, \mathbb{C}) \xrightarrow{\sim} H^\bullet(X, \mathbb{C})^G \tag{5.1.2}$$

on the Betti cohomology spaces.

Proof. As a corollary of the material in paragraph 5.3 of [Gro57] one can prove the following fact. Let K be a field of characteristic zero, T a Hausdorff space and G a finite group acting on T. Then the canonical map $H^\bullet(T/G, K) \to H^\bullet(T, K)^G$ is an isomorphism.

Therefore it remains to show that we have a homeomorphism $X(\mathbb{C})/G \xrightarrow{\sim} Z(\mathbb{C})$, where $Z = X/G$ is the *algebraic* quotient and $X(\mathbb{C})/G$ is the *topological* quotient of the action that is induced by G on $X(\mathbb{C})$.

Now let $q_{alg}\colon X \to Z$ denote the algebraic quotient map. By the construction of the complex topology it is obvious that $q_{alg}(\mathbb{C})\colon X(\mathbb{C}) \to Z(\mathbb{C})$ is continuous. From the universal property of the topological quotient map q_{top} we then find a continuous map β that makes the following diagram commute:

$$\begin{tikzcd} X(\mathbb{C}) \arrow[r, "q_{top}"] \arrow[dr, "q_{alg}(\mathbb{C})"'] & X(\mathbb{C})/G \arrow[d, "\beta"] \\ & Z(\mathbb{C}) \end{tikzcd}$$

It is easy to see that β is a bijection, since the fibers of q_{alg} are precisely the orbits of G.

It only remains to show that β is an open map. Since q_{top} is continuous and surjective, this follows if we can show that $q_{alg}(\mathbb{C})$ is open. This is indeed the case, due to a variation of the open mapping theorem. See paragraph 5.4 in [GR84]. But then β must be an open map as well, and this concludes the proof. □

By [Gro66, Theorem 1] the commuting property from proposition 5.1.3 also holds for the algebraic de Rham cohomology of a smooth affine $\mathbb{C}$-scheme X with a G-action such that X/G is again smooth. In proposition 3.4.6 we gave a sketch of an algebraic proof of this property. The algebraic proof has the advantage that it works over any base field K of characteristic zero. As an application we obtained a more direct proof that the map $H^{n-1}_{dR}(\Psi_K)$ from the proof of proposition 3.4.5 is bijective. We also refer to [Bri98, Theorem 1]. This result generalizes the statement of proposition 3.4.6, at least when the base field K is algebraically closed.

Next we consider a smooth proper $\mathbb{C}$-scheme X. In this situation we have an isomorphism $H^\bullet_{dR}(X) \cong H^\bullet(X, \mathbb{C})$, which follows from the holomorphic Poincaré lemma (combined with GAGA). By the isomorphism (5.1.2) one finds that the algebraic de Rham cohomology commutes with the action of a finite group G, at least if the quotient X/G is smooth. It seems likely that for such an X the commuting property

$$H^\bullet_{dR}(X/G) \xrightarrow{\sim} H^\bullet_{dR}(X)^G$$

can also be proved in a purely algebraic way, as was the case for smooth affine schemes.

In conclusion, it seems that the difficulty of question 5.1.1 comes from the following two facts:

i) There is not always a clear connection between rigid cohomology and lifted data in characteristic zero. This is in particular the case for weighted projective schemes, as we have explained in remark 3.4.2. Also see the remarks in paragraph 4.2.6.

ii) In general we cannot lift the situation to characteristic zero.

This second point is even true in the simplest case where X is a smooth affine scheme over a field k of characteristic $p > 0$. Indeed, write $X = \operatorname{Spec} \overline{A}$ and let A be the weak completion of $\overline{A}$. If $G \subset \operatorname{Aut}(\overline{A})$ is a finite group, then one knows that each automorphism $g \in G$ lifts to A. However, it is *a priori* not clear that the lifted set again forms a group under composition. There is the additional difficulty that one has to consider the continuous differentials $D^\bullet(A)$ on A, *not* the traditional differential forms $\Omega^\bullet_B$ on a smooth lift B of $\overline{A}$. This means that proposition 3.4.6 is not sufficient, even in cases where the group action lifts to the weak completions. Compare this with the question just below [MW68, Theorem 8.6], which does not seem to have been solved to this day.

The situation in proposition 3.4.5 is special because we can use theorem 1.2.2 (the Baldassarri-Chiarellotto theorem) to take care of the lifting step. Also, it is obvious that the G-action lifts to characteristic zero. However, this approach would fail for a general ramified cover.

Further comments

To end this paragraph we give some more clarifications about the results from [Gro57] that were used in the proof of proposition 5.1.3. It is explained just after [Gro57, Théorème 5.2.1] that the canonical map $H^\bullet(T/G, K) \to H^\bullet(T, K)^G$ is an isomorphism if the spectral sequences $I_2^{p,q}$ and $II_2^{p,q}$ from the Theorem degenerate at their second page. In the context of proposition 5.1.3 the spectral sequence $II_2^{p,q}$ is degenerate. Indeed, $H^i(G, V) = 0$ for a finite group G, a vector space V over a field K of characteristic zero and $i > 0$. Now proposition [Gro57, Proposition 5.2.3] says that $I_2^{p,q}$ is degenerate if the higher G-invariant pushforwards of the coefficient sheaf $\mathcal{F}$ vanish:

$$\mathbb{R}^i\psi_*^G(\mathcal{F}) = 0 \quad \text{for} \quad i > 0, \tag{5.1.3}$$

where ψ denotes the quotient map. The corollary of [Gro57, Théorème 5.3.1] shows that this is the case under the assumption "(D)", stated at the beginning of that paragraph. This assumption is always satisfied for a Hausdorff space T and a finite group G. One should also be careful that in [Gro57], the term *séparé* means *Hausdorff*; it should not be confused with the concept of separability for schemes. There is a warning that the assumption "(D)" is *not* true for an algebraic variety with its Zariski topology. At the end of paragraph 5.2 it is explained that for algebraic varieties, the condition (5.1.3) is guaranteed to hold when $\mathcal{F}$ is coherent and the quotient map is unramified.

This sheds some more light on the reason why the commuting property holds very generally for Hausdorff spaces, whereas for varieties one usually considers étale Galois covers. In general one should verify that the condition (5.1.3) holds.

It seems that this idea can be generalized for G-sheaves on any site. In fact, the proof of the Hochschild-Serre spectral sequence as presented in [Mil80, Theorem III.2.20] is very close in spirit to the material in [Gro57]. With some additional effort it may be possible to prove stronger commuting properties for étale cohomology, compare for example with [Got96, Proposition 3.6].

5.1.2 The weighted projective case

Choose a tuple of weights $\underline{w} = (w_1, \ldots, w_n)$ and consider a quasi-smooth weighted projective hypersurface $S_\infty \subset \mathbb{P}_k(\underline{w})$. At the beginning of chapter 3 we have explained that S_∞ can be seen as the quotient of a projective hypersurface $\widetilde{S}_\infty \subset \mathbb{P}_k^{n-1}$ by a finite Abelian group $G(\underline{w})$. If there is at least one $w_i > 1$ then the cover $\mathbb{P}_k^{n-1} \to \mathbb{P}_k(\underline{w})$ is ramified. In this situation we are not aware of any proof that the canonical map

$$H_{rig}^{n-1}(\mathbb{P}_k(\underline{w}) \setminus S_\infty) \longrightarrow H_{rig}^{n-1}(\mathbb{P}_k^{n-1} \setminus \widetilde{S}_\infty)^{G(\underline{w})} \tag{5.1.4}$$

is an isomorphism. However, there is no reason to believe that this would *not* be the case. For this reason we make the following conjecture.

Conjecture 5.1.4. *The map (5.1.4) is an isomorphism for any tuple of weights $\underline{w}$ and any quasi-smooth hypersurface S_∞, assuming that $\widetilde{S}_\infty$ is also smooth.*

For the Betti cohomology of a quasi-smooth weighted projective hypersurface over $\mathbb{C}$ we know, by proposition 5.1.3, that the canonical map

$$H^{n-1}(\mathbb{P}_{\mathbb{C}}(\underline{w}) \setminus S_\infty, \mathbb{C}) \longrightarrow H^{n-1}(\mathbb{P}^{n-1}_{\mathbb{C}} \setminus \widetilde{S}_\infty, \mathbb{C})^{G(\underline{w})} \tag{5.1.5}$$

is an isomorphism. In this situation there are some classical results that make the arrow (5.1.5) more concrete. See our overview in the proof of proposition 3.4.1. In remark 3.4.2 we observed that some of these complex analytical results do not have a direct counterpart in rigid cohomology. This comparison gives a more intuitive sense of the difficulty of conjecture 5.1.4.

In chapter 4 we proved that the code of the *Frobenius project* [dJ06] approximates the Frobenius action on the $G(\underline{w})$-invariant rigid cohomology $H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}$. A proof of conjecture 5.1.4 would justify the claim that this modification of the AKR algorithm computes the Frobenius action on $H^{n-1}_{rig}(\mathbb{P}_k(\underline{w}) \setminus S_\infty)$. See paragraph 4.2.6 for detailed explanations.

Another application of conjecture 5.1.4 is the computation of local rigid cohomology spaces by a "change of weights" argument, similar to what we used in paragraph 4.3.3 to determine the local cohomology of an A_j singularity. Indeed, consider a tuple $\underline{w} = (w_1, \ldots, w_n)$ satisfying the condition

$$\gcd(w_1, \ldots, w_n) = 1.$$

Following the procedure described in paragraph 1.3 of [Dol82], one obtains a tuple $\underline{w'} = (w'_1, \ldots, w'_n)$ satisfying

$$\gcd(w'_1, \ldots, \widehat{w'_i}, \ldots, w'_n) = 1$$

for all i and such that $\mathbb{P}_k(\underline{w}) \cong \mathbb{P}_k(\underline{w'})$. Under this isomorphism, a weighted projective hypersurface $S_\infty = Z(g) \subset \mathbb{P}_k(\underline{w})$ will be transformed into another hypersurface $S'_\infty = Z(g') \subset \mathbb{P}_k(\underline{w'})$. Now let $Y = Z(g) \subset \mathbb{A}^n_k$ resp. $Y' = Z(g') \subset \mathbb{A}^n_k$ denote the weighted homogeneous singularities defined by g resp. by g'. Also assume that $\widetilde{S'}_\infty$ is smooth. By combining theorem 3.1.11 and conjecture 5.1.4 we obtain the following identifications:

$$\begin{tikzcd}
H^{n-1}_{rig}(\mathbb{P}_k(\underline{w}) \setminus S_\infty) \arrow[r, "\cong"] \arrow[d, "\cong"] & H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})} \arrow[r, "\cong"] & H^n_{rig,\{0\}}(Y) \\
H^{n-1}_{rig}(\mathbb{P}_k(\underline{w'}) \setminus S'_\infty) \arrow[r, "\cong"] & H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S'}_\infty)^{G(\underline{w'})} \arrow[r, "\cong"] & H^n_{rig,\{0\}}(Y')
\end{tikzcd}$$

We conclude that the local rigid cohomology spaces $H^n_{rig,\{0\}}(Y)$ and $H^n_{rig,\{0\}}(Y')$ are Frobenius-isomorphic.

Recall that we have already used a similar technique in paragraph 4.3.3 to determine the Frobenius action on the local cohomology of a singularity of type A_j. This case is special because one finds $\underline{w'} = (1, \ldots, 1)$. From this it follows that $\mathbb{P}_k(\underline{w}) \setminus S_\infty$ is smooth, and as a result the canonical map (5.1.4) is injective. Conjecture 5.1.4 is then reduced to verifying that the dimensions are the same on both sides. See the proof of proposition 4.3.10 for details.

However, the weighted projective complements $\mathbb{P}_k(\underline{w}) \setminus S_\infty$ and $\mathbb{P}_k(\underline{w'}) \setminus S'_\infty$ will be singular in general. In this case we are not aware of any proof that the map (5.1.4) is injective. So in general the weight-change technique really depends on conjecture 5.1.4.

As noted above, we can combine theorem 3.1.11 and conjecture 5.1.4 to obtain an isomorphism

$$H^n_{rig,\{0\}}(Y) \xrightarrow{\sim} H^{n-1}_{rig}(\mathbb{P}_k(\underline{w}) \setminus S_\infty). \tag{5.1.6}$$

However, this isomorphism still relies on the assumption that $\widetilde{S}_\infty$ is smooth. In order to remove the smoothness assumption on $\widetilde{S}_\infty$, one can try to follow the proof of proposition 3.4.3, which deals with an analogous result for Betti cohomology. For this we should at least solve another instance of question 5.1.1. The weighted projective complement $\mathbb{P}_k(\underline{w}) \setminus S_\infty$ may be identified with the quotient of the affine Milnor fiber $S \setminus S_\infty$ by its monodromy action. This motivates the following conjecture:

Conjecture 5.1.5. *The quotient map $S \setminus S_\infty \to \mathbb{P}_k(\underline{w}) \setminus S_\infty$ induces an isomorphism*

$$H^{n-1}_{rig}(\mathbb{P}_k(\underline{w}) \setminus S_\infty) \xrightarrow{\sim} H^{n-1}_{rig}(S \setminus S_\infty)^{\langle \zeta_d \rangle}.$$

Again, we refer to remark 3.4.2 for explanations about the obstacles to proving this conjecture. We also note that conjecture 5.1.5 is not quite sufficient to prove an isomorphism (5.1.6) without the assumption that $\widetilde{S}_\infty$ is smooth. We also need to show that $H^i_{rig}(S \setminus S_\infty) = 0$ for $i \notin \{0, n-1\}$, which would follow from a Poincaré duality theorem for quasi-smooth weighted projective hypersurfaces. Compare this with the proof of proposition 3.4.3.

5.1.3 An unpublished result of Kloosterman

To close this section we present a result that was communicated by Remke Kloosterman.

Proposition 5.1.6. *Consider a quasi-projective k-scheme X with an action of a finite group G. Assume moreover that G is Abelian and that the order of G is not divisible by the characteristic of k. Then the quotient map $X \to X/G$ induces an isomorphism*

$$H^\bullet_{rig,c}(X/G) \xrightarrow{\sim} H^\bullet_{rig,c}(X)^G \tag{5.1.7}$$

on the rigid cohomology with compact supports.

Note that since the quotient map $X \to X/G$ is finite (hence proper), the rigid cohomology with compact supports behaves in a contravariant manner.

The statement of proposition 5.1.6 should be compared with that of [Ete08, Théorème IV.4.2]. The latter result states that $H^{\bullet}_{rig,c}$ commutes with G when $X \to X/G$ is a finite étale Galois cover (for compact supports the quotient X/G is not assumed to be smooth).

The idea behind the proof of proposition 5.1.6 is as follows. First one reduces to the case where G is a cyclic group of prime order. Then one can proceed by induction on the dimension of X. The induction step is based on the long exact sequence (1.2.22) for rigid cohomology with compact supports. It seems that the existence of this sequence makes the situation significantly easier than for the regular rigid cohomology.

Prof. Kloosterman has communicated this result as a fix for an issue in the paper [Klo07]. More specifically, proposition 5.1.6 is meant to replace [Klo07, Proposition 3.4]. The latter proposition is equivalent to the claim that for a general smooth affine k-scheme U with an action of a finite group G the canonical map $H^{\bullet}_{rig}(U/G) \to H^{\bullet}_{MW}(U)^G$ is an isomorphism. The proof of this proposition is only a sketch. The idea is to express the rigid cohomology of U/G in terms of a realization $(U/G \subset Y/G \subset P/G)$, where $(U \subset Y \subset P)$ is a realization of U.

There are two problems with this approach. Firstly, it is not obvious that the G-action on U extends to a G-action on a realization. Secondly, the formal scheme P/G may be singular. It is then not obvious that carrying out the usual constructions w.r.t. to the frame $(U/G \subset Y/G \subset P/G)$ will give the rigid cohomology $H^{\bullet}_{rig}(U/G)$. Note however that when U/G is smooth and when the quotient map $U \to U/G$ is an étale Galois cover, the approach suggested by Kloosterman works. Indeed, the same approach is worked out in detail in the proof of [Ete08, Théorème IV.4.2].

The claim of [Klo07, Proposition 3.4] is applied to the scheme $\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty$ and the group $G(\underline{w})$. In other words: the paper relies on conjecture 5.1.4.

However, most proofs in [Klo07] go through without modification if one replaces [Klo07, Proposition 3.4] with proposition 5.1.7 below. Only the statement of [Klo07, Theorem 3.10] needs to be reformulated as follows: Poincaré duality holds for $\mathbb{P}_k(\underline{w}) \setminus S_\infty$ if and only if conjecture 5.1.4 holds for $S_\infty \subset \mathbb{P}_k(\underline{w})$.

Proposition 5.1.7. *Consider a weighted projective hypersurface $S_\infty \subset \mathbb{P}_k(\underline{w})$. Then there is a Frobenius-equivariant isomorphism*

$$H^{n-1}_{rig,c}(\mathbb{P}_k(\underline{w}) \setminus S_\infty)^\vee \xrightarrow{\sim} [H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}](n-1).$$

The $\vee$ on the left-hand side denotes the dual space, the $(n-1)$ on the right-hand side signals a Frobenius twist.

Proof. By proposition 5.1.6 there is a Frobenius-equivariant isomorphism

$$H^{n-1}_{rig,c}(\mathbb{P}_k(\underline{w}) \setminus S_\infty) \xrightarrow{\sim} H^{n-1}_{rig,c}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}.$$

Since $\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty$ is smooth we can use Poincaré duality. It is easy to see that the resulting isomorphism

$$H^{n-1}_{rig,c}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty) \xrightarrow{\sim} H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)(n-1)^\vee$$

is also $G(\underline{w})$-equivariant. The proposition follows. $\square$

Remark 5.1.8. Note that the same argument would hold for any smooth quasi-projective scheme X and any group G that satisfies the conditions of proposition 5.1.6. If X/G is also smooth, then we even obtain an isomorphism $H^\bullet_{rig}(X/G) \xrightarrow{\sim} H^\bullet_{rig}(X)^G$. In this way one finds another class of pairs (X, G) for which the question 5.1.1 has an affirmative answer.

As a consequence of the proposition above, and assuming that $\widetilde{S}_\infty$ is smooth, the zeta function of $\mathbb{P}_k(\underline{w}) \setminus S_\infty$ may be written as:

$$\begin{aligned} &Z(\mathbb{P}_k(\underline{w}) \setminus S_\infty, T) \\ &\quad = (1 - q^{n-1}T)^{-1} \cdot \det\left(\mathrm{Id} - \mathrm{Fr}T \mid H^{n-1}_{rig,c}(\mathbb{P}_k(\underline{w}) \setminus S_\infty)\right)^{(-1)^n} \\ &\quad = (1 - q^{n-1}T)^{-1} \cdot \det\left(\mathrm{Id} - q^{n-1}\mathrm{Fr}^{-1}T \mid H^{n-1}_{rig}(\mathbb{P}^{n-1}_k \setminus \widetilde{S}_\infty)^{G(\underline{w})}\right)^{(-1)^n} \end{aligned}$$

Several propositions in [Klo07] implicitly make use of this expression of the zeta function. We have also used this formula at the end of paragraph 4.2.7, to prove that the characteristic polynomial of Frobenius belongs to $\mathbb{Z}[T]$. As another application we can give a partial answer to the technical difficulties that we discussed in paragraph 4.2.6. Proposition 5.1.7 shows that the modified AKR algorithm from section 4.2 (or equivalently, the algorithm from [dJ06]) can be used to approximate a cohomological object related to the weighted projective scheme $\mathbb{P}_k(\underline{w}) \setminus S_\infty$. If one calculates with high enough precision then the zeta function can be recovered. On the other hand, there still seems to be no way to get around the assumption that $\widetilde{S}_\infty$ is smooth.

5.2 Dimca's method in rigid cohomology

The paper [Dim90a] contains a topological method for the computation of the Betti cohomology $H^{n-1}(\mathbb{P}^n_{\mathbb{C}} \setminus X, \mathbb{C})$, for $X \subset \mathbb{P}^n_{\mathbb{C}}$ a hypersurface with weighted homogeneous singular points. Also see chapter 6 of [Dim92], where the same method is covered in more detail.

In this section we discuss which parts of Dimca's method can be carried over to rigid cohomology. We also argue that some of the techniques used in [Dim90a] have no clear analogues in rigid cohomology.

The set-up in [Dim90a] is as follows. Fix a complex hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$ that has only isolated weighted homogeneous singularities. Denote the singular locus by $\Sigma \subset X$. Since X is singular, the Betti cohomology $H^{n-1}(\mathbb{P}^n_{\mathbb{C}} \setminus X, \mathbb{C})$ is in general not zero. Also see corollary 5.3.2 in the next section. It is then possible to identify the Betti cohomology $H^{n-1}(\mathbb{P}^n_{\mathbb{C}} \setminus X, \mathbb{C})$ with the cokernel of a certain map

$$H^n(\mathbb{P}^n_{\mathbb{C}} \setminus X, \mathbb{C}) \longrightarrow H^n_{\Sigma}(X, \mathbb{C}) \tag{5.2.1}$$

that can be constructed from standard long exact sequences. The main idea in [Dim90a] is that this map is concrete enough to be able to compute the cokernel. To see this, one first applies the isomorphism

$$H^n(\mathbb{P}^n_{\mathbb{C}} \setminus X, \mathbb{C}) \xrightarrow{\sim} H^n_{dR}(\mathbb{P}^n_{\mathbb{C}} \setminus X)$$

that is described in [Gro66, Theorem 1]. For this one uses the fact that $\mathbb{P}^n_{\mathbb{C}} \setminus X$ is smooth affine of dimension n. In paragraph 1.2.3 we have seen that for a smooth X, the algebraic de Rham cohomology of $\mathbb{P}^n_{\mathbb{C}} \setminus X$ can be described in terms of the differential forms

$$\frac{A\,\Omega}{F^{n-t}}$$

where F is the defining equation of X, $d = \deg F$ and $\deg A = (n-t) \cdot d - n - 1$. The relations among these differential forms, in the case where X is a smooth hypersurface, are described in [Gri69]. See also paragraph 1.2.3 of the introduction. In our setting X is singular, and some of the results in [Gri69] are no longer true. However, even for X singular, the map

$$\begin{array}{rcl} \mathbb{C}[x_0, \ldots, x_n]_{(n-t)\cdot d-n-1} & \longrightarrow & H^n_{dR}(\mathbb{P}^n_{\mathbb{C}} \setminus X) \\ A & \mapsto & \dfrac{A\,\Omega}{F^{n-t}} \end{array}$$

is surjective for $t = 0$. This is due to a relation between the images for $t \geq 0$ (the so-called *polar filtration*) and the Hodge filtration $F^{\bullet}_H$ on $H^n_{dR}(\mathbb{P}^n_{\mathbb{C}} \setminus X)$, see [Dim90a, Proposition 1.2]. For a general $t \geq 0$, the image contains the subset $F^{t+1}_H H^n_{dR}(\mathbb{P}^n_{\mathbb{C}} \setminus X)$.

The cokernel of the map (5.2.1) is now equal to the cokernel of a map

$$\mathbb{C}[x_0, \ldots, x_n]_{(n-t)\cdot d-n-1} \longrightarrow H^n_{\Sigma}(X, \mathbb{C}) \tag{5.2.2}$$

for a suitable choice of $t \geq 0$. This is expressed in [Dim90a, Theorem 2.4]. Usually one is interested in the *maximal* value of t for which the cokernel of (5.2.2) completely determines the cohomology $H^{n-1}(\mathbb{P}^n_{\mathbb{C}} \setminus X, \mathbb{C})$. It is known that, if X has only weighted homogeneous singularities, then the Hodge filtration on $H^n_\Sigma(X, \mathbb{C})$ coincides with the polar filtration[2]. In this case the maximal value for t can be determined. This is explained in more detail in [Dim90a, Corollary 2.6].

The central idea of the paper [Dim90a] is that if one has a detailed understanding of the local Betti cohomology $H^n_\Sigma(X, \mathbb{C})$, then the map (5.2.2) can be described in a quite explicit way. This results in a method to compute the dimension (and with some luck, the mixed Hodge structure) of $H^{n-1}(\mathbb{P}^n_\mathbb{C} \setminus X, \mathbb{C})$.

For example, assume that n is even and consider an X that has only ordinary double points. Then the arrow (5.2.2) may be identified with the map

$$\begin{aligned} \mathbb{C}[x_0, \dots, x_n]_{(n-t)\cdot d-n-1} &\longrightarrow H^n_\Sigma(X, \mathbb{C}) = \bigoplus_{P \in \Sigma} W \\ A \quad &\mapsto \oplus_{P \in \Sigma}\, A(\widetilde{P}) \cdot v \end{aligned} \tag{5.2.3}$$

where $t = \frac{n}{2}$. In the equation above, v is a basis vector of the one-dimensional $\mathbb{C}$-space W, which is the local cohomology space of an ordinary double point. The points $\widetilde{P} \in \mathbb{C}^{n+1}$ are representatives of the projective points $P \in \Sigma$. These representatives are completely determined by the choice of v. See [Dim90a, Proposition 3.3] for details. With the formula above, the cohomology $H^{n-1}(\mathbb{P}^n_\mathbb{C} \setminus X, \mathbb{C})$ can be understood as the defect of a linear system in $\mathbb{C}[x_0, \dots, x_n]_{(n-t)\cdot d-n-1}$.

The explicit formulation of the map (5.2.2) for an X that has only A_1-singularities is a consequence of [Dim90a, Corollary 2.5]. Here it is shown that the kernel of (5.2.2) is given by certain linear conditions that can be determined explicitly if one has a good understanding of the local cohomology space $H^n_\Sigma(X, \mathbb{C})$. The idea of the proof is that the map (5.2.1) is essentially the restriction of a global differential form to a small analytic neighbourhood of a singular point $P \in \Sigma$. If one represents a global differential form by a polynomial $A \in \mathbb{C}[x_0, \dots, x_n]_{(n-t)\cdot d-n-1}$, then this translates to a linear condition that is satisfied by A at the point P. It seems possible to explicitly write down the relevant linear conditions for any hypersurface X having weighted homogeneous singularities, even though this is not completely worked out in [Dim90a].

5.2.1 Rigid cohomology

We now discuss which parts of Dimca's method can be carried over to rigid cohomology, and for which parts this is not immediately possible.

One important ingredient in [Dim90a] is the following fact about weighted homogeneous singularities. Assume that $Y = Z(g)$ is a weighted homogeneous

[2] The local cohomology carries a polar filtration coming from *local* differential forms.

hypersurface singularity, defined by an analytic germ g around the origin. Then there is an isomorphism

$$H^n_{\{0\}}(Y, \mathbb{C}) \xrightarrow{\sim} H^n(B_\varepsilon \setminus Y, \mathbb{C}) \tag{5.2.4}$$

were B_ε is a small ball around the origin. Our proposition 3.2.1 should be seen as a partial analogue in rigid cohomology. Indeed, if the complex germ g above is chosen to be a weighted homogeneous polynomial then the isomorphism (5.2.4) above also holds with B_ε replaced by $\mathbb{A}^n_{\mathbb{C}}$. The need for a small analytic neighbourhood only arises when one replaces g by a contact equivalent germ that is *not* a weighted homogeneous polynomial. When working over a base field k of positive characteristic, it is obvious that there is no isomorphism similar to (5.2.4) for a general local equation g. But we can use theorem 2.1.1 to find a weighted homogeneous equation, which can then be used together with proposition 3.2.1.

Now assume that $X \subset \mathbb{P}^n_k$ is a projective hypersurface with a non-empty zero-dimensional singular locus $\Sigma \subset X$. We assume that all the singular points of X are weighted homogeneous. Then it is still true that the rigid cohomology $H^{n-1}(\mathbb{P}^n_k \setminus X)$ may be identified with the cokernel of a certain map

$$H^n_{rig}(\mathbb{P}^n_k \setminus X) \longrightarrow H^n_{rig,\Sigma}(X). \tag{5.2.5}$$

The proof is quite straightforward. In [Dim90a] the construction of the map (5.2.1) uses certain long exact sequences from topology. For the construction of (5.2.5) we need to replace these with the standard long exact sequences from rigid cohomology. We break up the proof into several lemmas.

Proposition 5.2.1. *Take $X \subset \mathbb{P}^n_k$ as above, with $n \geq 3$. Then for each $i \geq 2$ there are isomorphisms*

$$H^i_{rig,c}(X \setminus \Sigma) \xrightarrow{\sim} H^i_{rig,c}(X)$$

and

$$H^i_{rig,c}(\mathbb{P}^n_k \setminus \Sigma) \xrightarrow{\sim} H^i_{rig,c}(\mathbb{P}^n_k).$$

The spaces $H^1_{rig,c}(X \setminus \Sigma)$ and $H^1_{rig,c}(\mathbb{P}^n_k \setminus \Sigma)$ are both of dimension $|\Sigma| - 1$. Also,

$$\dim H^0_{rig,c}(X \setminus \Sigma) = \dim H^0_{rig,c}(\mathbb{P}^n_k \setminus \Sigma) = 0.$$

Proof. The first statement follows immediately by considering the long exact sequence (1.2.22) for the pairs $\Sigma \subset X$ and $\Sigma \subset \mathbb{P}^n_k$.

Now consider the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 \longrightarrow & H^0_{rig,c}(\mathbb{P}^n_k \setminus \Sigma) & \longrightarrow & H^0_{rig,c}(\mathbb{P}^n_k) & \longrightarrow & H^0_{rig,c}(\Sigma) & \longrightarrow & H^1_{rig,c}(\mathbb{P}^n_k \setminus \Sigma) & \longrightarrow 0 \\
& \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & \\
0 \longrightarrow & H^0_{rig,c}(X \setminus \Sigma) & \longrightarrow & H^0_{rig,c}(X) & \longrightarrow & H^0_{rig,c}(\Sigma) & \longrightarrow & H^1_{rig,c}(X \setminus \Sigma) & \longrightarrow 0
\end{array}
$$

The zero at the end of the bottom row is justified by the fact that

$$H^i_{rig,c}(\mathbb{P}^n_k \setminus X) = 0 \text{ for } 0 \leq i \leq n-1.$$

We assumed that $n \geq 3$ and therefore $H^2_{rig,c}(\mathbb{P}^n_k \setminus X) = 0$. From this we find that $H^1_{rig,c}(X) = 0$.

A similar argument shows that the arrow β is an isomorphism. Of course γ is the identity, and it follows that α and δ are isomorphisms as well.

Using the theory of cycle classes from [Pet03] it is easy to prove that the arrow $H^0_{rig,c}(\mathbb{P}^n_k) \to H^0_{rig,c}(\Sigma)$ is not zero. Therefore it must be injective, which shows that $H^0_{rig,c}(\mathbb{P}^n_k \setminus \Sigma)$ (and also $H^0_{rig,c}(X \setminus \Sigma)$) is zero.

The dimension of $H^1_{rig,c}(\mathbb{P}^n_k \setminus \Sigma)$ (and also $H^1_{rig,c}(X \setminus \Sigma)$) must be equal to $|\Sigma| - 1$. □

Proposition 5.2.2. *Take $X \subset \mathbb{P}^n_k$ as above, with $n \geq 3$. Assume moreover that X has only isolated weighted homogeneous singularities (satisfying the assumptions from definition 3.1.7). Then the following properties hold:*

i) The Gysin map $H^{i-2}_{rig}(X \setminus \Sigma)(-1) \to H^i_{rig}(\mathbb{P}^n_k \setminus \Sigma)$ is an isomorphism for $i \notin \{0, n, n+1\}$.

ii) For $i \notin \{0, n-1, n\}$ we have $H^i_{rig}(\mathbb{P}^n_k \setminus X) = 0$.

Proof. The scheme $\mathbb{P}^n_k \setminus X$ is smooth affine of dimension n, so we know that $H^i_{rig}(\mathbb{P}^n_k \setminus X) = 0$ for $i > n$. Combining this with the Gysin sequence of the pair $X \setminus \Sigma \subset \mathbb{P}^n_k \setminus \Sigma$ we find that the Gysin map $H^{i-2}_{rig}(X \setminus \Sigma)(-1) \to H^i_{rig}(\mathbb{P}^n_k \setminus \Sigma)$ is an isomorphism for $i > n+1$.

Recall that the Gysin map $H^{i-2}_{rig}(X \setminus \Sigma)(-1) \to H^i_{rig}(\mathbb{P}^n_k \setminus \Sigma)$ is dual to the canonical map $H^{2n-i}_{rig,c}(\mathbb{P}^n_k \setminus \Sigma) \to H^{2n-i}_{rig,c}(X \setminus \Sigma)$ on rigid cohomology with compact supports. By proposition 5.2.1 it now remains to show that the maps $H^i_{rig,c}(\mathbb{P}^n_k) \to H^i_{rig,c}(X)$ are isomorphisms for $n < i < 2n$. Since X is proper we may also drop the compact supports.

But under our assumptions we have $\dim H^i_{rig}(X) = \dim H^i_{rig}(\mathbb{P}^n_k)$ in this range, see proposition 5.3.5 below. This step is the only real difference with the proof of proposition 3.5.1. The result now follows from the fact that the maps $H^i_{rig}(\mathbb{P}^n_k) \to H^i_{rig}(X)$ are nonzero whenever $i < 2n$ is even. This is proved using the theory of cycle classes. More precisely, [Pet03, Proposition 6.4] implies that the dual maps $H^i_{rig}(X)^\vee \to H^i_{rig}(\mathbb{P}^n_k)^\vee$ are nonzero for $i < 2n$ even. □

As a result of this proposition we find that $H^{n-1}_{rig}(\mathbb{P}^n_k \setminus X)$ may be identified wit the primitive part of $H^{n-2}_{rig}(X \setminus \Sigma)(-1)$. This should be compared with [Dim90a, Lemma 2.2]. In fact, one could make an alternative, more algebraic, proof of [Dim90a, Lemma 2.2] by using the vanishing property of corollary 5.3.2 below.

With proposition 5.2.2 in place we can construct the map (5.2.5) using only the Gysin sequence for $X \setminus \Sigma \subset \mathbb{P}^n_k \setminus \Sigma$ and the long exact sequence with supports (1.2.18) for the pairs $\Sigma \subset X$ and $\Sigma \subset \mathbb{P}^n_k$. In the end one finds that $H^{n-1}_{rig}(\mathbb{P}^n_k \setminus X)$ may be identified with the cokernel of (5.2.5). The proof is formally the same as the argument just below [Dim90a, Lemma 2.2].

This brings us to a central idea in [Dim90a] that does *not* have a direct counterpart for rigid cohomology. Just below [Dim90a, Lemma 2.2] it is argued that the map (5.2.1) is induced by restricting a global differential form to a small neighbourhood of a singular point $P \in \Sigma$. To see this property one really needs the topological constructions in the proof of [Dim90a, Lemma 2.2]. The algebraic proof of [Dim90a, Lemma 2.2] that we outlined above does not tell one how to explicitly compute the maps (5.2.1) and (5.2.2). In the construction of (5.2.5) we were forced to follow an arithmetic approach, and we see no good way to calculate the cokernel.

An equivalent way of expressing this difficulty is as follows. A global differential form on $\mathbb{P}^n_{\mathbb{C}} \setminus X$ can be expressed as a rational function whose denominator is a power of F. Under the map (5.2.2) this global differential form will be transformed into a local differential form at a singular point, which can be expressed in terms of a local weighted homogeneous equation g. More specifically, it is the isomorphism (5.2.4) that will make the local equation g appear. This property is expressed in [Dim90a, Proposition 1.5]. In the arithmetic setting we have to work with étale neighbourhoods, and we should use the isomorphism from theorem 2.1.1 instead. This isomorphism is too complicated to be understood in concrete terms. Theorem 3.1.11 does allow us to write the local cohomology in terms of a weighted homogeneous local equation g. However, there is no clear connection with the global differential forms on $\mathbb{P}^n_k \setminus X$. This shows that for rigid cohomology the situation is much more difficult than in the topological setting.

In the end we are left with the question:

Question 5.2.3. *Can Dimca's method be adapted to rigid cohomology?*

5.2.2 A calculation

We do not know if Dimca's method generally works for rigid cohomology. However, if $X \subset \mathbb{P}^n_k$ (with n even) has only ordinary double points then it is still more or less possible to write down the map (5.2.3). Indeed, for any lift

$\mathcal{X} \subset \mathbb{P}^n_{\mathcal{V}}$ we have a specialization map

$$H^n_{dR}(\mathbb{P}^n_K \setminus \mathcal{X}_K) \to H^n_{rig}(\mathbb{P}^n_k \setminus X).$$

In general we cannot expect this map to be surjective. On the other hand, surjectivity seems plausible if the singular locus $\mathcal{S}_K \subset \mathcal{X}_K$ is a lift of the singular locus $\Sigma \subset X$. Some results in this direction have been obtained in [Klo08]. So after composing with (5.2.5) we can identify $H^{n-1}_{rig}(\mathbb{P}^n_k \setminus X)$ with the cokernel of a map

$$K[x_0, \ldots, x_n]_{(n-t)\cdot d-n-1} \longrightarrow H^n_{dR}(\mathbb{P}^n_K \setminus \mathcal{X}_K) \longrightarrow H^n_{rig,\Sigma}(X).$$

If question 5.2.3 has an affirmative answer then one expects, since X has only ordinary double points, that the map above can be written explicitly as

$$A \mapsto \oplus_{P \in \mathcal{S}_K} A(\widetilde{P}) \cdot v,$$

for $t = \frac{n}{2}$, similarly to (5.2.3).

Let us try to carry out this computation for the hypersurface from example 3.1.3 over the base field $k = \mathbb{F}_q$ with (say) $q = 11$. In this case k has a primitive fifth root of unity, and X has 125 ordinary double points. The numbers b_i from equation (3.1.2) satisfy

$$\prod_{i=1}^{4} b_i = 5 \in \mathbb{F}_{11},$$

which is a square. So for each $P \in \Sigma$ the local cohomology space $H^4_{rig,\{P\}}(X)$ is one-dimension with Frobenius acting as the multiplication by q^2. This follows from proposition 4.3.6.

Now take $K = \mathbb{Q}_{11}$. We lift the hypersurface by considering the equation F from example 3.1.3 as an element of $R = K[x_0, \ldots, x_4]$. The singular locus of the resulting hypersurface $\mathcal{X}_K$ is given by

$$\mathcal{S}_K = \left\{ (\zeta^{a_0} : \zeta^{a_1} : \ \ldots : \zeta^{a_4}) \mid \sum_{i=0}^{4} a_i \equiv 0 \mod 5 \right\}$$

where $\zeta \in \mathbb{Q}_{11}$ is a primitive fifth root of unity. In the end we consider the map (with $t = 2$):

$$\begin{aligned} R_5 &\longrightarrow \bigoplus_{P \in \mathcal{S}_K} \mathbb{Q}_{11} \cdot v \\ A &\mapsto \oplus_{P \in \mathcal{S}_K} A(\widetilde{P}) \end{aligned} \tag{5.2.6}$$

The choice of the basis vector v and of the representatives $\widetilde{P} \in (\mathbb{Q}_{11})^5$ do not affect the cokernel, so we leave them implicit.

By calculating the cokernel of (5.2.6) we can make the following conjecture: The rigid cohomology space $H^3_{rig}(\mathbb{P}^4_k \setminus X)$ has dimension 24, with Frobenius acting as the multiplication by q^2. This is consistent with [CdlORV03], in which the zeta function of $\mathbb{P}^4_k \setminus X$ is calculated in a very different way. Therefore we ask:

Question 5.2.4. *Can the computation above be justified in a rigorous way? If so, can it be generalized?*

5.3 Vanishing properties of $H^\bullet_{rig}(\mathbb{P}^n_k \setminus X)$

Let K be a field of characteristic zero and consider a projective hypersurface $X = Z(F) \subset \mathbb{P}^n_K$ whose singular locus is of dimension $0 \leq m < n-1$. It is a well-known fact that the algebraic de Rham cohomology satisfies

$$H^i_{dR}(\mathbb{P}^n_K \setminus X) = 0 \quad \text{for} \quad 0 < i < n - m - 1. \tag{5.3.1}$$

One proof of this fact is presented in [Dim92, Corollary 6.2.22]. Alternatively, see [Dim90b, Corollary 1.13]. The idea is to compare the de Rham cohomology with the Koszul complex $K^\bullet_F$ of the partial derivatives $\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}$. One then concludes by using certain vanishing properties of the Koszul complex $K^\bullet_F$. More specifically, there is the following lemma.

Proposition 5.3.1. *With the notations above, one has $H^i(K^\bullet_F) = 0$ for $i < n - m$.*

Proof. Write $R = K[x_0, \ldots, x_n]$ and consider the ideal

$$J = \left(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n} \right) \subset R.$$

According to [Eis95, Theorem 17.4], we have that $H^i(K^\bullet_F) = 0$ for $i < \operatorname{depth}(J)$. So it remains to show that $\operatorname{depth}(J) = n - m$.

Now consider the *codimension* of J, which is given by

$$\operatorname{codim}(J) = \min \left\{ \operatorname{codim}(P) \mid P \text{ is an isolated prime} \in \operatorname{Ass}(J) \right\}.$$

By definition, the codimension of a prime ideal P is the maximal length of a strictly increasing sequence of primes ending in P. Since R is a Cohen-Macaulay ring, we have $\operatorname{depth}(J) = \operatorname{codim}(J)$. See Theorems 18.7 and 18.9 in

[Eis95]. Now we compute:

$$\begin{aligned}
\dim R/J &= \max\{\dim R/P \mid P \text{ is an isolated prime} \in \mathrm{Ass}(J)\} \\
&= \dim R - \min\{\mathrm{codim}(P) \mid P \text{ is an isolated prime} \in \mathrm{Ass}(J)\} \\
&= \dim R - \mathrm{codim}(J) \\
&= \dim R - \mathrm{depth}(J)
\end{aligned}$$

Since $\dim R = n+1$ and $\dim R/J = m+1$ we indeed find that

$$\mathrm{depth}(J) = n - m.$$

This finishes the proof. □

Corollary 5.3.2. *For $0 < i < n-m-1$ one has $H^i_{dR}(\mathbb{P}^n_K \setminus X) = 0$.*

Proof. This is explained in paragraph 6.2 of [Dim92]. □

Note that the proof of proposition 5.3.1 is slightly simpler than the argument used in [Dim92, Proposition 6.2.21]. It has the additional advantage that it proves the vanishing property of $H^i_{dR}(\mathbb{P}^n_K \setminus X)$ over any base field of characteristic zero, rather than just over $\mathbb{C}$.

Now consider the case where $X = Z(F) \subset \mathbb{P}^n_k$ is a projective hypersurface over a field k of positive characteristic. Again we let m denote the dimension of the singular locus. By analogy to the above, we make the following conjecture about the vanishing of rigid cohomology:

Conjecture 5.3.3. *Using the notation from above, we have*

$$H^i_{rig}(\mathbb{P}^n_k \setminus X) = 0 \quad \text{for} \quad 0 < i < n-m-1.$$

The difficulty is that the theorem of Baldassarri-Chiarellotto is not applicable, due to the fact that X is singular. As a result, there is no clear relation to the de Rham cohomology $H^i_{dR}(\mathbb{P}^n_K \setminus \mathcal{X}_K)$ of a lifted hypersurface $\mathcal{X}$. We really need to reason on the level overconvergent structures, and this makes the problem quite difficult.

One idea is to follow a Dworkian approach and use the theory from [BB04]. We first explain that one can obtain an alternative proof of (5.3.1) by using [BB04, Corollary 5.7], which gives a relation between de Rham cohomology and a certain complex $\mathcal{L}^\bullet$. After applying this result, it suffices to show that

$$H^i(\mathcal{L}^\bullet) = 0 \text{ for } i < n-m. \tag{5.3.2}$$

The notation $\mathcal{L}^\bullet$ is taken from Chapter 7 of [Mon70]. This is the same complex as in [BB04, Definition 4.1], which appears in [BB04, Corollary 5.7]. It it shown in [Mon70, Theorem 8.1] that the complex $\mathcal{L}^\bullet$ is isomorphic to a somewhat easier complex $\mathcal{L}^\bullet(F)$. One can then put a suitable grading on $\mathcal{L}^\bullet(F)$, similarly to what is done in paragraph 6.2 of [Dim92]. By considering the resulting spectral sequence it is possible to prove the following: property (5.3.2) holds if the (classical) Koszul complex $K_F^\bullet$ satisfies $H^i(K_F^\bullet) = 0$ for $i < n - m$. The precise computation is very similar to the proof of [Dim92, Theorem 6.2.9]. One then concludes with proposition 5.3.1.

The statement of [BB04, Corollary 5.7] also gives a relation between rigid cohomology and a certain *overconvergent* complex $L^\bullet$. By following the proof of [Mon70, Theorem 8.1] it is again possible to prove that $L^\bullet$ is isomorphic to a slightly easier complex $L^\bullet(F)$. The precise definition of $L^\bullet(F)$ should be clear by analogy. Then [BB04, Corollary 5.7] allows us to conclude that conjecture 5.3.3 holds if $H^i(L^\bullet(F)) = 0$ for $i < n - m$. Unfortunately, it seems that the overconvergent nature of $L^\bullet(F)$ makes it much harder to study than its counterpart $\mathcal{L}^\bullet(F)$. The problem is that the complex $L^\bullet(F)$, which essentially consists of overconvergent power series, does not admit any reasonable grading. This leads us to the following conjecture:

Conjecture 5.3.4. *We have*

$$H^i(L^\bullet(F)) = 0 \quad \textit{for} \quad i < n - m.$$

To end this section we will prove conjecture 5.3.3 for a certain class of hypersurfaces having only isolated singularities (i.e., we take $m = 0$). The proof of this fact has been started in proposition 5.2.2, but there is one more lemma left.

Proposition 5.3.5. *Let $X \subset \mathbb{P}_k^n$ (with $n \geq 3$) be a projective hypersurface with only isolated singularities. Denote its singular locus by Σ. Assume moreover that the following conditions hold:*

i) $H^i_{rig,\Sigma}(X) = 0$ *for* $n < i < 2n - 2$

ii) $\dim H^{2n-2}_{rig,\Sigma} = |\Sigma|$

Then we have an equality of dimensions

$$\dim H^i_{rig}(X) = \dim H^i(\mathbb{P}_k^n) \quad \textit{for} \quad n < i < 2n.$$

Proof. Since $2n - 1 > 2 \cdot \dim X$ we have that

$$\dim H^{2n-1}_{rig}(X) = 0 = \dim H^{2n-1}_{rig}(\mathbb{P}_k^n).$$

By proposition 5.2.1 and Poincaré duality we have an isomorphism

$$H^{2n-2}_{rig}(X) = H^{2n-2}_{rig,c}(X) \xrightarrow{\sim} H^{2n-2}_{rig,c}(X \setminus \Sigma) \xrightarrow{\sim} H^0_{rig}(X \setminus \Sigma).$$

But by proposition 3.2.5 we know that $H^0_{rig}(X \setminus \Sigma) \cong H^0_{rig}(X)$. This yields:

$$\dim H^{2n-2}_{rig}(X) = 1 = \dim H^{2n-2}_{rig}(\mathbb{P}^n_k).$$

Now consider the long exact sequence with supports (1.2.18) for the pair $\Sigma \subset X$. It follows from assumption i) that the map $H^i_{rig}(X) \to H^i_{rig}(X \setminus \Sigma)$ is an isomorphism for $n < i < 2n-3$. In the proof of proposition 5.2.2 we already showed that

$$\dim H^i_{rig}(X \setminus \Sigma) = \dim H^i_{rig}(\mathbb{P}^n_k \setminus \Sigma) = \dim H^i_{rig}(\mathbb{P}^n_k)$$

in this range. It remains to show that $H^{2n-3}_{rig}(X) = 0$. For this we consider the exact sequence

$$0 \longrightarrow H^{2n-3}_{rig}(X) \longrightarrow H^{2n-3}_{rig}(X \setminus \Sigma) \longrightarrow H^{2n-2}_{rig,\Sigma}(X) \longrightarrow H^{2n-2}_{rig}(X) \longrightarrow 0$$

Here we have used proposition 5.2.1 together with Poincaré duality, which shows that $H^{2n-2}_{rig}(X \setminus \Sigma) = 0$. In a similar way we have that

$$\dim H^{2n-3}_{rig}(X \setminus \Sigma) = |\Sigma| - 1.$$

By computing the alternating sum of the dimensions in the exact sequence above, and combining this with assumption ii), we indeed find that $H^{2n-3}_{rig}(X) = 0$. ☐

Corollary 5.3.6. *Conjecture 5.3.3 holds for hypersurfaces X that satisfy the conditions of proposition 5.3.5.*

Proof. We already proved this in proposition 5.2.2. ☐

Note that the conditions of proposition 5.3.5 are satisfied for hypersurfaces X that have only weighted homogeneous singularities. Indeed, this follows from theorem 3.1.11, combined with the fact that the space $H^i_{rig,\Sigma}(X)$ splits into a direct sum $\bigoplus_{x \in \Sigma} H^i_{rig,\{x\}}(X)$.

5.4 Proper base change and the long exact sequence of a resolution

In this section we discuss the possibility that the rigid cohomology of a singular scheme X may be understood in terms of a resolution of singularities. More specifically, we ask ourselves if a modification of k-schemes $f \colon \widetilde{X} \to X$ gives rise to a long exact sequence in rigid cohomology.

It is to be expected that such a long exact sequence exists, since this is known to be the case for étale cohomology. See for example the proof that is presented in [Klo12, Proposition 2.3]. This proof mostly consists of very general topos-theoretic constructions. This suggests that for rigid cohomology we should try to use the topos-theoretic formulation from [LS11].

We now give a precise statement of the conjecture, together with a partial proof. The difference with the proof of [Klo12, Proposition 2.3] is that we only use the *extension by zero* $j_!$ of an open immersion j, which is defined on any site. To our knowledge the *(higher) direct image with compact supports* $f_!$ (resp. $R^i f_!$) of a compactifiable morphism f has not yet been defined for the overconvergent site.

Conjecture 5.4.1. *Let $f\colon \widetilde{X} \to X$ be a modification of a variety X over a field k of characteristic $p > 0$. In other words, f is a proper birational morphism. Let $Z \subset X$ be a closed subset such that the restriction*

$$f\colon \widetilde{X} \setminus f^{-1}(Z) \to X \setminus Z$$

is an isomorphism. Then there is a long exact sequence (with $\widetilde{Z} := f^{-1}(Z)$):

$$\ldots \to H^i_{rig}(X) \to H^i_{rig}(\widetilde{X}) \oplus H^i_{rig}(Z) \to H^i_{rig}(\widetilde{Z}) \to H^{i+1}_{rig}(X) \to \ldots \tag{5.4.1}$$

Partial proof. Write $U = X \setminus Z$ and $\widetilde{U} = \widetilde{X} \setminus \widetilde{Z}$. We denote p_X the structural morphism of X, and we use similar notation for the schemes $\widetilde{X}$, Z, $\widetilde{Z}$, U, $\widetilde{U}$. The rigid cohomology of X can then be expressed as $(\mathbb{R}p_{X*})\mathcal{O}^\dagger_X$, where $\mathcal{O}^\dagger_X$ is an object of the overconvergent topos $X_{\mathrm{AN}^\dagger}$. See paragraph 1.2.5 of the introduction for more details. Similar notations hold for the other schemes involved.

We now have the following two Cartesian diagrams

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "\widetilde{i}"] \arrow[d, "g"'] & \widetilde{X} \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd} \qquad \begin{tikzcd} \widetilde{U} \arrow[r, "\widetilde{j}"] \arrow[d, "\alpha"', "\cong"] & \widetilde{X} \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

where $i, \widetilde{i}$ are closed immersions and $j, \widetilde{j}$ are open immersions. By general facts of topos theory we have two short exact sequences

$$\begin{gathered} 0 \longrightarrow j_!\mathcal{O}^\dagger_U \longrightarrow \mathcal{O}^\dagger_X \longrightarrow i_*\mathcal{O}^\dagger_Z \longrightarrow 0 \\ 0 \longrightarrow \widetilde{j}_!\mathcal{O}^\dagger_{\widetilde{U}} \longrightarrow \mathcal{O}^\dagger_{\widetilde{X}} \longrightarrow \widetilde{i}_*\mathcal{O}^\dagger_{\widetilde{Z}} \longrightarrow 0 \end{gathered} \tag{5.4.2}$$

that induce long exact sequences

$$
\begin{tikzcd}
\ldots \arrow[r] & (\mathbb{R}^m p_{X*}) j_! \mathcal{O}_U^\dagger \arrow[r] \arrow[d, "\phi_m"] & (\mathbb{R}^m p_{X*}) \mathcal{O}_X^\dagger \arrow[r] \arrow[d] & (\mathbb{R}^m p_{Z*}) \mathcal{O}_Z^\dagger \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & (\mathbb{R}^m p_{\widetilde{X}*}) \widetilde{j}_! \mathcal{O}_{\widetilde{U}}^\dagger \arrow[r] & (\mathbb{R}^m p_{\widetilde{X}*}) \mathcal{O}_{\widetilde{X}}^\dagger \arrow[r] & (\mathbb{R}^m p_{\widetilde{Z}*}) \mathcal{O}_E^\dagger \arrow[r] & \ldots
\end{tikzcd}
\tag{5.4.3}
$$

See exposé IV, paragraph 14 of [SGA4] for the definitions of $j_!$ and $\widetilde{j}_!$ and of the sequences (5.4.2). The vertical arrows in (5.4.3) correspond to the canonical map (2.1.7) on rigid cohomology. To see this, use the isomorphisms

$$f^* j_! \mathcal{O}_U^\dagger \xrightarrow{\sim} \widetilde{j}_! \alpha^* \mathcal{O}_U^\dagger = \widetilde{j}_! \mathcal{O}_{\widetilde{U}}^\dagger$$

and

$$f^* i_* \mathcal{O}_Z^\dagger \xrightarrow{\sim} \widetilde{i}_* g^* \mathcal{O}_Z^\dagger = \widetilde{i}_* \mathcal{O}_{\widetilde{Z}}^\dagger.$$

The first one of these isomorphisms is explained in [SGA4, Lemme XVII.5.1.2], the second one is a general fact about closed subtoposes.

Since α is an isomorphism we may write $j_! \mathcal{O}_U^\dagger = j_! (\mathbb{R}\alpha_*) \mathcal{O}_{\widetilde{U}}^\dagger$. By doing so we may understand ϕ_m as applying $\mathbb{R}^m p_{X*}$ to an arrow

$$j_! (\mathbb{R}^m \alpha_*) \mathcal{O}_{\widetilde{U}}^\dagger \longrightarrow (\mathbb{R}^m f_*) \widetilde{j}_! \mathcal{O}_{\widetilde{U}}^\dagger \tag{5.4.4}$$

that is defined similarly as in [SGA4, XVII.5.1.5].

If the map (5.4.4) is an isomorphism, then we can conclude the proof using a diagram-chasing argument that is formally the same as in [Har75, Proposition II.4.3]. □

We do not know if the map (5.4.4) is an isomorphism. This is the only point that is incomplete in the proof above.

It is known that on the étale site $X_{ét}$ on a scheme X, the map corresponding to (5.4.4) is indeed an isomorphism. However, the proof relies on the proper base change theorem in a subtle way. The details can be found in the proof of [SGA4, Lemme XVII.5.1.6].

It seems that there does not yet exist a general proper base change theorem for the overconvergent site. The only proper base change theorem that we know of is due to Etesse, see [Ete08, Théorème IV.3.1]. The proof of this Theorem uses the classical definition of rigid cohomology. One of the major difficulties is that one needs to make sure that the derived pushforward of an overconvergent isocrystal is again overconvergent. As a result, the proof of [Ete08, Théorème IV.3.1] works under very strong assumptions. In particular, X is assumed to be smooth.

So it seems that in order to prove conjecture 5.4.1, one needs an answer to the following question:

Question 5.4.2. *Does the overconvergent site admit a proper base change theorem that is strong enough to prove that the map (5.4.4) is an isomorphism?*

One long-standing open problem about varieties over an algebraically closed field k of positive characteristic goes as follows: does a k-variety X admit a resolution of singularities? By a *resolution* we mean a modification $f\colon \widetilde{X} \to X$ where $\widetilde{X}$ is smooth. Over a field of characteristic zero such a resolution is guaranteed to exist by the famous theorem of Hironaka. Over a field of positive characteristic the existence of a resolution is currently only known for varieties of dimension at most 3. See the article [Hau10] for an overview of the relevant results.

The best known general result in positive characteristic is the *alteration theorem* from [dJ96]. This result has found many applications, some of which were discussed in this thesis. However, a smooth alteration is still weaker than a resolution, in the sense that every modification is also an alteration.

Nevertheless, if one finds a resolution $\widetilde{X} \to X$ for a particular variety X then the long exact sequence (5.4.1) could be useful to understand the rigid cohomology $H^\bullet_{rig}(X)$ in terms of the cohomology space $H^\bullet_{rig}(\widetilde{X})$. Since $\widetilde{X}$ is smooth, one expects its rigid cohomology to be easier to understand.

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 9. August 2016 David Ouwehand